

Optimal stopping of Gauss-Markov bridges

with applications to American options

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PROBLEM

The mathematical problem

Solving the **Optimal Stopping Problem** (OSP)

$$V(t, x) = \sup_{\tau \leq T-t} \mathbb{E}_{t,x} [G(t + \tau, X_{t+\tau})]$$

for processes $(X_t)_{t \in [0, T]}$ that degenerate at the horizon T

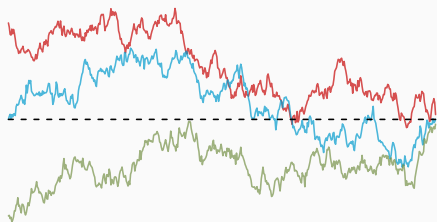
The financial problem

Under strong beliefs about an asset's price at a terminal time:

- Optimally selling/buying a derivative
- Optimally exercising an American option

Terminal pinning point scenarios:

- Stock-pinning effect^{1,2}
- Perishable assets/commodities³
- Forecasts/predictions
- Mispriced assets



Gauss–Markov bridges:

Conditioning a non-degenerated GM process to degenerate at maturity

- Brownian bridge
- Ornstein–Uhlenbeck bridge (application to pair trading)

¹ Krishnan et al. (2001). The effect of stock pinning upon option prices. *Risk*

² Golez et al. (2012). Pinning in the S&P 500 futures. *J. Financ. Econ.*

³ Boyce (1970). Stopping rules for selling bonds. *Bell J. Econom. Manage. Sci.*

Equivalent definitions of GM processes¹⁻³

- Gaussian and Markovian at the same time
- Diffusion representation: $dX_t = (\alpha(t) + \beta(t)X_t) dt + \gamma(t)dB_t$

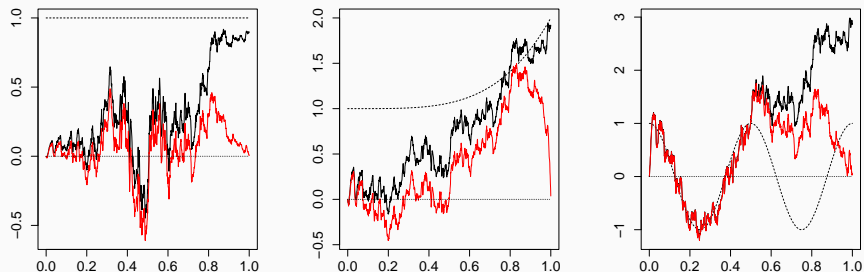


Figure: Different paths of GM processes (—) and bridges (—) derived from them, changing the pulling level (—) — β/α

¹ Mehr et al. (1965). Certain properties of Gaussian processes and their first-passage times. *J. R. Stat. Soc. Ser. B Methodol.*

² Borisov (1983). On a criterion for Gaussian random processes to be Markovian. *Theory Probab. Its Appl.*

³ Buonocore et al. (2013). On some time-non-homogeneous linear diffusion processes and related bridges. *Sci. Math. Jpn.*

OPTIMAL STOPPING THEORY

Solving an OSP

Solving the OSP $V(t, x) = \sup_{\tau \leq T-t} \mathbb{E}_{t,x} [G(t + \tau, X_{t+\tau})]$ consists in finding:

- a tractable characterization of V (**value function**) and
- an **Optimal Stopping Time** (OST), i.e. τ^* such that $V(t, x) = \mathbb{E}_{t,x} [G(t + \tau^*, X_{t+\tau^*})]$

Theorem^{1,2}

Define:

- **Stopping set** $\mathcal{D} := \{(t, x) : V(t, x) = G(t, x)\}$
- **First entry to \mathcal{D}** : $\tau_{\mathcal{D}}$

Assume that:

- $\mathbb{E}_{t,x} [\sup_t |G(t, X_t)|] < \infty$, for all t, x
- V and G are continuous

Then:

- **Supermartingale characterization of the value function:**
 - ◇ **Supermartingale:** $\mathbb{E}_{t,x} [V(t + \tau, X_{t+\tau})] \geq V(t, x)$ for all τ
 - ◇ **Domination:** $V(t, x) \geq G(t, x)$
 - ◇ **Smallest:** if W is a supermartingale that dominates G , then $W \geq V$
- **Optimality of $\tau_{\mathcal{D}}$:**
 - ◇ **Optimality:** $\tau_{\mathcal{D}}$ is an OST if $\tau_{\mathcal{D}} < \infty$ a.s.
 - ◇ **Smallest:** if τ^* is an OST, then $\tau_{\mathcal{D}} \leq \tau^*$ a.s.

¹ Dynkin (1963). The optimum choice of the instant for stopping a Markov process. *Sov. Math. Dokl.*

² Peskir et al. (2006). *Optimal Stopping and Free-Boundary Problems*. Birkhäuser.

The free-boundary problem¹

Under some regularity conditions, V and \mathcal{D} solve the **Free-Boundary Problem** (FBP)

$$\begin{aligned} \mathbb{L}V &= 0 && \text{on } \mathcal{C} := \{(t, x) : V(t, x) > G(t, x)\} = \mathcal{D}^c \\ V &= G && \text{on } \mathcal{D} \\ \partial_x V &= \partial_x G && \text{on } \partial\mathcal{D} \text{ (smooth-fit condition)} \end{aligned}$$

\mathbb{L} is the **infinitesimal generator** of X

Challenges

- GMBs are **time-inhomogeneous** processes with **explosive drifts**
- **Finite horizon** T

¹ Peskir et al. (2006). Optimal Stopping and Free-Boundary Problems. Birkhäuser.

OPTIMAL STOPPING OF GAUSS–MARKOV BRIDGES

- 📄 D’Auria, B., García-Portugués, E., and Guada, A. (2020) Discounted optimal stopping of a Brownian bridge, with application to American options under pinning. *Mathematics*, 8(7): 1159. doi: [10.3390/math8071159](https://doi.org/10.3390/math8071159)
- 📄 Azze, A., D’Auria, B., and García-Portugués, E. (2024) Optimal stopping of an Ornstein–Uhlenbeck bridge. *Stochastics Processes and Their Applications*. doi: [10.1016/j.spa.2024.104342](https://doi.org/10.1016/j.spa.2024.104342)
- 📄 Azze, A., D’Auria, B., and García-Portugués, E. (to appear, 2025) Optimal stopping of Gauss–Markov bridges. *Advances in Applied Probability*. arXiv: [2211.05835](https://arxiv.org/abs/2211.05835)

Optimal Stopping Problem

$$V(t, x) = \sup_{\tau \leq T-t} \mathbb{E}_{t,x} [X_{t+\tau}], \quad X_t \text{ is a Gauss–Markov Bridge (GMB)}$$

Related problems

- Brownian bridge^{1–3}

Contributions

- ✓ Characterization of the OSB via an integral equation
- ✓ Numerical analysis of the OSB
- ✓ Upper bound of the OSB via a comparison argument
- ✓ Lipschitz continuity of the OSB
- ✓ Time-change technique

¹ Shepp (1969). Explicit solutions to some problems of optimal stopping. *Ann. Math. Statist.*

² Ekström et al. (2009). Optimal stopping of a Brownian bridge. *J. Appl. Probab.*

³ De Angelis et al. (2020). Optimal stopping for the exponential of a Brownian bridge. *J. Appl. Probab.*

Proposition (Gauss-Markov bridges' equivalent definitions)

For the stochastic process $X = \{X_t\}_{t \in [0, T]}$, the following statements are equivalent:

- (i) X results after conditioning a time-continuous, non-degenerated GM process to start at $(0, x)$ and end at (T, z)
- (ii) X admits the representation

$$\begin{cases} X_t = \alpha(t) + \beta_T(t) \left((z - \alpha(T))\gamma_T(t) + \left(B_{\gamma_T(t)} + \frac{x - \alpha(0)}{\beta_T(0)} \right) \right), & t \in [0, T] \\ X_T = z \end{cases}$$

$\alpha : [0, T] \rightarrow \mathbb{R}$, $\beta_T : [0, T] \rightarrow \mathbb{R}_+$, and $\gamma_T : [0, T] \rightarrow \mathbb{R}_+$ satisfy regularity conditions

REFORMULATION OF THE OSP

Time transformation: $s = \gamma_T(t)$
Space transformation: $y = \mu_T(x), c = \mu_T(z)$

GMB

$\{X_t\}_{t \in [0, T]}, X_0 = x, X_T = z$

$X_t = G_c(s, Y_s)$

BM

$\{Y_s\}_{s \in \mathbb{R}_+}, Y_0 = y$

Original OSP

$V(t, x) := \sup_{\tau \leq T-t} \mathbb{E}_{t, x} [X_{t+\tau}]$

$V(t, x) = W(s, y)$

Transformed OSP

$W(s, y) := \sup_{\sigma} \mathbb{E}_{s, y} [G_c(s + \sigma, Y_{s+\sigma})]$

Original OST

$\tau^*(t, x)$

$t + \tau^*(t, x) = \gamma_T^{-1}(s + \sigma^*(s, y))$

Transformed OST

$\sigma^*(s, y)$

SOLUTION METHOD FOR W

Regularity of the OSB

$$\mathcal{D} = \{(s, y) : y \geq b(s)\}$$

b is bounded

Regularity of W

W is LC in bounded sets

W is increasing and convex in x

W is $C^{1,2}$ in \mathcal{D} and in \mathcal{C}

$$\left(\partial_t + \frac{1}{2}\partial_{xx}\right)W = 0 \text{ in } \mathcal{C}$$

$\partial_x W$ and $\partial_t W$ are explicitly bounded

b is LC in closed intervals

Law of the iterated logarithm

Smooth-fit condition ($\partial_x V = \partial_x G$ on $\partial\mathcal{D}$)

Itô's formula to $W(s+u, Y_{s+u})$

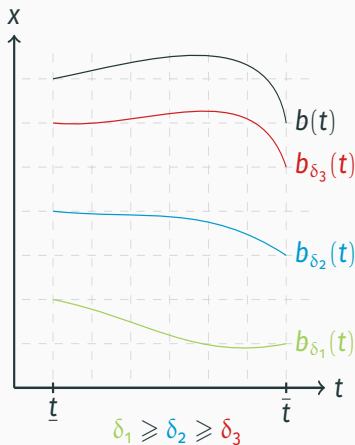
Free-boundary equation

THE OSB IS LIPSCHITZ CONTINUOUS IN ANY CLOSED INTERVAL¹

- Define $H := W - G$ and $I := [\underline{t}, \bar{t}] \subset \mathbb{R}$
- Using properties of H , we construct, for $\delta > 0$ small, $b_\delta : I \rightarrow \mathbb{R}$ such that $H(t, b_\delta(t)) = \delta$
- We prove that $b_\delta(t) \xrightarrow{\delta \rightarrow 0} b(t)$
- The implicit function theorem yields

$$b'_\delta(t) = -\partial_t H(t, b_\delta(t)) / \partial_x H(t, b_\delta(t))$$

- We found bounds for $\partial_t H(t, b_\delta(t))$ and $\partial_x H(t, b_\delta(t))$, uniformly in δ and $t \in I$
- The Arzelà–Ascoli theorem implies that $b_\delta(t)$ converges to $b(t)$, uniformly in δ and $t \in I$



To bound $\partial_t H(t, b_\delta(t))$, the Brownian motion representation of X_t is key

¹ De Angelis et al. (2019). On Lipschitz continuous optimal stopping boundaries. *SIAM J. Control Optim.*

Theorem

- **OST:** $\tau^*(t, x) := \inf\{u \in [0, T - t] : X_{t+u} \geq b(t + u)\}$
- **OSB:** The unique solution, up to regularity conditions, of the integral equation

$$b_{T,z}(t) = z - \int_t^T K(t, b_{T,z}(t), u, b_{T,z}(u)) du$$

- **Value function:** $V(t, x) = z - \int_t^T K(t, x, u, b(u)) du$
- **Kernel:**

$$K(t_1, x_1, t_2, x_2) = v_{t_2}(t_1) \frac{\beta'_T(t_2)}{\beta_T(t_2)} \Phi\left(\frac{x_2 - m_{t_2}(t_1, x_1)}{v_{t_2}(t_1)}\right) + \bar{\Phi}\left(\frac{x_2 - m_{t_2}(t_1, x_1)}{v_{t_2}(t_1)}\right) \\ \times \left(\alpha'(t_2) + (m_{t_2}(t_1, x_1) - \alpha(t_2)) \frac{\beta'_T(t_2)}{\beta_T(t_2)} + (z - \alpha(T)) \beta_T(t_2) \gamma'_T(t_2)\right)$$

- ◇ ϕ and $\bar{\Phi}$ are the density and survival functions of a standard normal r.v.
- ◇ $m_{t_2}(t_1, x_1)$ and $v_{t_2}(t_1)$ are **explicit**

Problem: The FBE is **not analytically solvable**

Numerical solution

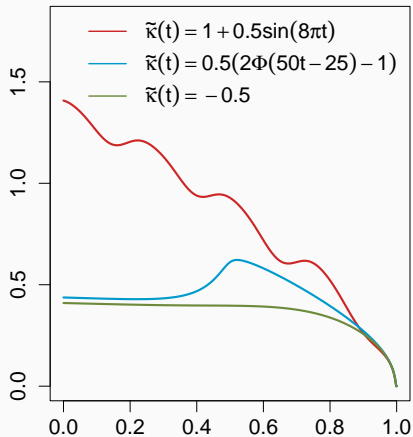
- Discretization of time: $0 = t_1 < \dots < t_N = T$
- Discretization of the FBE + Fixed-point (Picard's iteration) algorithm

$$b^{(k)}(t_i) := z - \sum_{j=i+1}^{N-1} K(t_i, b^{(k-1)}(t_i), t_j, b^{(k-1)}(t_j))$$

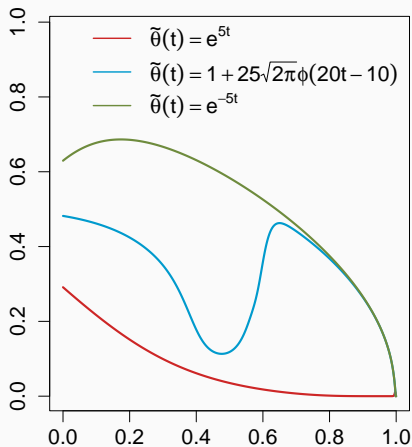
$$b^{(0)}(t) := b(T) = z, \quad \forall t \in [0, T]$$

- Last term of the Riemann sum excluded because **the kernel explodes at maturity**

$$X_t = \tilde{X}_t \mid \tilde{X}_T = 0, \quad d\tilde{X}_t = \tilde{\theta}(t)(\tilde{\kappa}(t) - \tilde{X}_t) dt + \tilde{v}(t) dB_t, \quad \tilde{v} \equiv 1$$



Varying $\tilde{\kappa}$ with $\tilde{\theta} \equiv 3$



Varying $\tilde{\theta}$ with $\tilde{\kappa} \equiv -1$

REAL-DATA STUDY

BB vs GBM: PERFORMANCE COMPARISON STUDY

$$\text{OSP: } V(t, x) = \sup_{\tau \leq T-t} \mathbb{E}_{t,x} [e^{-\lambda\tau} (A - X_{t+\tau})^+]$$

X_t is a BB, $X_T = A$ is the strike price, $\lambda > 0$ is the discount rate

Scenario: Stock-pinning (pinning point near strike price)

Goal: Compare the BB vs the classic GBM¹ in varying stock-pinning scenarios

Data

- Put options of Apple (8905) and IBM (4833) within January 2011 – September 2018
- Stock price divided by strike price to standardize
- Pinning deviance: $\pi := |X_T - 1|$

Performance

- Relative mean profit: $\frac{BB(p) - GBM(p)}{GBM(p)}$
- $BB(p)$ ($GBM(p)$) is the profit generated by the BB (GBM) averaged over all options with $\pi \leq p$

¹ Peskir (2005). On the American option problem. *Math. Finance*.

BB vs GBM: RESULTS

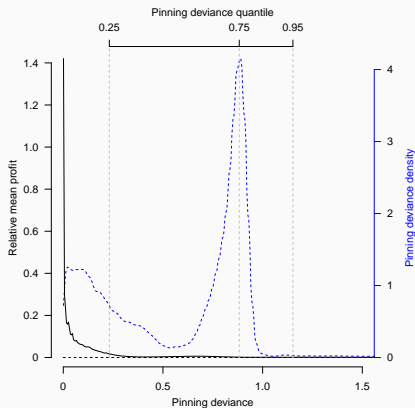


Figure: **Apple**

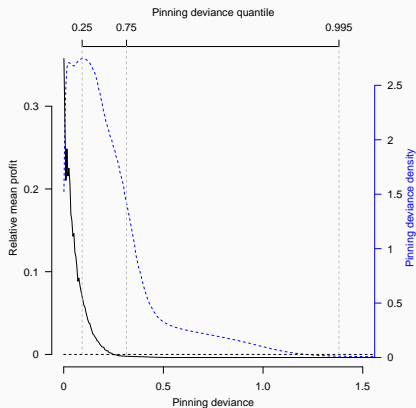


Figure: **IBM**

- BB outperforms GBM specially for low pinning deviance
- As the pinning deviance increases the advantage fades away
- The advantage offsets the losses even when using most of the options

FUTURE WORK

- Non-Gaussian Markovian bridges
- Random pinning point
- Relaxation of the OSB's smoothness to obtain the solution
- Prove the convergence of the fixed-point algorithms

Thanks!