CUNEF UNIVERSIDAD

Optimal stopping of Gauss–Markov bridges

Abel G. Azze

abel.guada@cunef.edu

CUNEF Universidad, Madrid

4th International Workshop on
Stochastic Processes and Their Applications

Universidad de Zaragoza, Rio de Zaragoza, Spain

10 July, 2024

Joint work with
Bernardo D'Auria (University of Padua) and Eduardo García Portugués (Universidad Carlos III de Madrid)

Project PID2020-116694GB-100 granted by Ministerio de Ciencia e Innovación

Motivation

When to exercise an American call option?

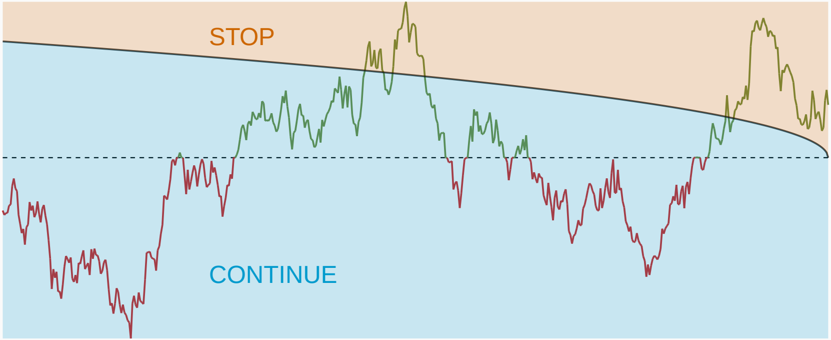
PROFIT

LOSS

American call option

The right to buy the stock at any time before *maturity* for the *strike price*

When to exercise an American call option?



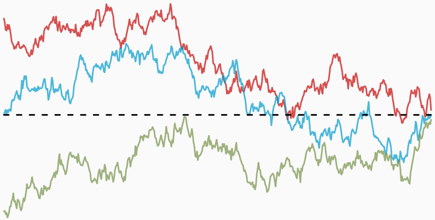
American call option

The right to buy the stock at any time before *maturity* for the *strike price*

Terminal pinning point:

The process degenerates at maturity

- Stock-pinning effect[1,2]
- Perishable assets/commodities[3]
- Forecasts/predictions
- Mispriced assets



Gauss–Markov bridges:

Conditioning a non-degenerated GM process to degenerate at maturity

- Brownian bridge
- Ornstein–Uhlenbeck bridge

[1] Krishnan et al. (2001). The effect of stock pinning upon option prices. *Risk*

[2] Golez et al. (2012). Pinning in the S&P 500 futures. *J. Financ. Econ.*

[3] Boyce (1970). Stopping rules for selling bonds. *Bell J. Econom. Manage. Sci.*

Gauss–Markov processes and bridges derived from them

Equivalent definitions of GM processes[1–3]

- Gaussian and Markovian at the same time
- Brownian motion representation: $X_t = a(t) + b(t)B_{h(t)}$
- Diffusion representation: $\mathrm{d}X_t = (\alpha(t) + \beta(t)X_t)\,\mathrm{d}t + \gamma(t)\mathrm{d}B_t$

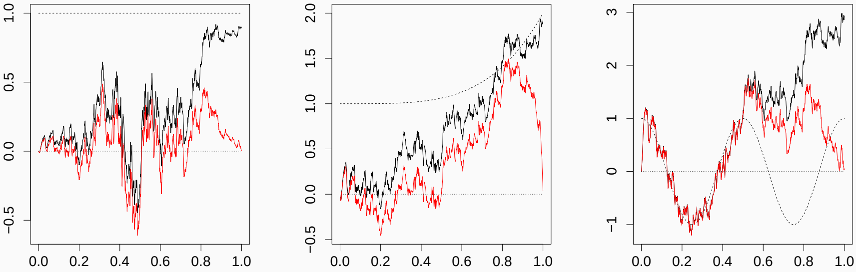


Figure: Different paths of GM processes (—) and bridges (—) derived from them, changing the pulling level (– –) $-\beta\,/\,\alpha$

[1] Mehr et al. (1965). Certain properties of Gaussian processes and their first-passage times. *J. R. Stat. Soc. Ser. B Methodol.*
[2] Borisov (1983). On a criterion for Gaussian random processes to be Markovian. *Theory Probab. Its Appl.*
[3] Buonocore et al. (2013). On some time-non-homogeneous linear diffusion processes and related bridges. *Sci. Math. Jpn.*

Structure

1. Motivation
2. Optimal stopping theory
3. Optimal stopping of Gauss–Markov bridges
4. Conclusions

Optimal stopping theory

Definition (Optimal Stopping Problem)

Let:

- $X = (X_t)_{t \in [0,T]}$ be a $\mathbb{R}^d$-valued, time-homogeneous and time-continuous Markov process on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_x)$, with $\mathbb{P}_x(X_0 = x) = 1$ for $x \in \mathbb{R}^d$
- $G : \mathbb{R}^d \to \mathbb{R}$ (gain function)

The Optimal Stopping Problem (OSP)

$$V(x) := \sup_{\tau \in [0,T]} \mathbb{E}_x \left[G(X_\tau) \right],$$

consists in finding

- a tractable characterization of V (value function) and
- an Optimal Stopping Time (OST), i.e. τ^* such that $V(x) = \mathbb{E}_x \left[G(X_{\tau^*}) \right]$ (if it exists)

Remark: time-inhomogeneous processes and time-dependent gain functions can be brought back to the standard case by considering the time-space process (t, X_t)

Characterization of V and τ^*

Theorem[1,2]

Define:

- Stopping set $\mathcal{D} := \{x : V(x) = G(x)\}$
- First entry to $\mathcal{D}$ $\tau_{\mathcal{D}} := \inf(t : X_t \in \mathcal{D})$

Assume that:

- $\mathbb{E}_x [\sup_t |G(X_t)|] < \infty$, for all x
- V and G are continuous

Then:

- Supermartingale characterization of the value function:
  - Supermartingale: $\mathbb{E}_x [V(X_\tau)] \geqslant V(x)$ for all τ
  - Domination: $V(x) \geqslant G(x)$
  - Smallest: if W is a supermartingale that dominates G, then $W \geqslant V$
- Optimality of $\tau_{\mathcal{D}}$:
  - Optimality: $\tau_{\mathcal{D}}$ is an OST if $\tau_{\mathcal{D}} < \infty$ a.s.
  - Smallest: if τ^* is an OST, then $\tau_{\mathcal{D}} \leqslant \tau^*$ a.s.

[1] Dynkin (1963). The optimum choice of the instant for stopping a Markov process. *Sov. Math. Dokl.*

[2] Peskir et al. (2006). Optimal Stopping and Free-Boundary Problems. Birkhäuser.

The free-boundary problem[1]

Under some regularity conditions, V and $\mathcal{D}$ solve the Free-Boundary Problem (FBP)

$$\begin{aligned} \mathbb{L}V &= 0 && \text{on } \mathcal{C} := \{x : V(x) > G(x)\} = \mathcal{D}^c \\ V &= G && \text{on } \mathcal{D} \\ \partial_x V &= \partial_x G && \text{on } \partial\mathcal{D} \text{ (smooth-fit condition)} \end{aligned}$$

$\mathbb{L}$ is the infinitesimal generator of X

The FBP inherits its **dimension** from that of the stochastic process:

- Dimension 1: The "guess-and-verify" method becomes tractable
- Higher dimension: Harder to solve the FBP (e.g., finite horizon, time-inhomogeneity, time-dependent gain)

[1] Peskir et al. (2006). Optimal Stopping and Free-Boundary Problems. Birkhäuser.

OPTIMAL STOPPING OF GAUSS–MARKOV BRIDGES

D'Auria, B., García-Portugués, E., and Guada, A. (2020) Discounted optimal stopping of a Brownian bridge, with application to American options under pinning. *Mathematics*, 8(7): 1159. doi: 10.3390/math8071159

Azze, A., D'Auria, B., and García-Portugués, E. (2024) Optimal stopping of an Ornstein–Uhlenbeck bridge. *Stochastics Processes and Their Applications*. doi: 10.1016/j.spa.2024.104342

Azze, A., D'Auria, B., and García-Portugués, E. (to apear, 2025) Optimal stopping of Gauss–Markov bridges. *Advances in Applied Probability*. arXiv: 2211.05835

OSP

$$V(t,x) = \sup_{\tau \leqslant T-t} \mathbb{E}_{t,x}\left[X_{t+\tau}\right], \quad X_t \text{ is a Gauss–Markov bridge}$$

Related problems

- Brownian bridge[1–3]

Main contributions

✓ Characterization of the OSB via an integral equation

✓ Numerical analysis of the OSB

✓ Upper bound of the OSB via a comparison argument

✓ Lipschitz continuity of the OSB

✓ Time-change technique

---

[1] Shepp (1969). Explicit solutions to some problems of optimal stopping. *Ann. Math. Statist.*

[2] Ekström et al. (2009). Optimal stopping of a Brownian bridge. *J. Appl. Probab.*

[3] De Angelis et al. (2020). Optimal stopping for the exponential of a Brownian bridge. *J. Appl. Probab.*

Proposition (Gauss–Markov bridges' equivalent definitions)

For the stochastic process $X = \{X_t\}_{t\in[0,T]}$, the following statements are equivalent:

(*i*) X results after conditioning a time-continuous, non-degenerated GM process to start at $(0, x)$ and end at (T, z)

(*ii*) X admits the representation

$$\begin{cases} X_t = \alpha(t) + \beta_T(t)\left((z-\alpha(T))\gamma_T(t) + \left(B_{\gamma_T(t)} + \dfrac{x-\alpha(0)}{\beta_T(0)}\right)\right), \; t \in [0, T) \\ X_T = z \end{cases}$$

$\alpha : [0, T] \to \mathbb{R}$, $\beta_T : [0, T] \to \mathbb{R}_+$, and $\gamma_T : [0, T) \to \mathbb{R}_+$ satisfy regularity conditions

Reformulation of the OSP

Time transformation: $s = \gamma_T(t)$

Space transformation: $y = \mu_T(x)$, $c = \mu_T(z)$

GMB: $\{X_t\}_{t \in [0,T]},\ X_0 = x,\ X_T = z$

$X_t = G_c(s, Y_s)$ ⟷

BM: $\{Y_s\}_{s \in \mathbb{R}_+},\ Y_0 = y$

Original OSP: $V(t,x) := \sup_{\tau \leqslant T-t} \mathbb{E}_{t,x}\left[X_{t+\tau}\right]$

$V(t,x) = W(s,y)$ ⟷

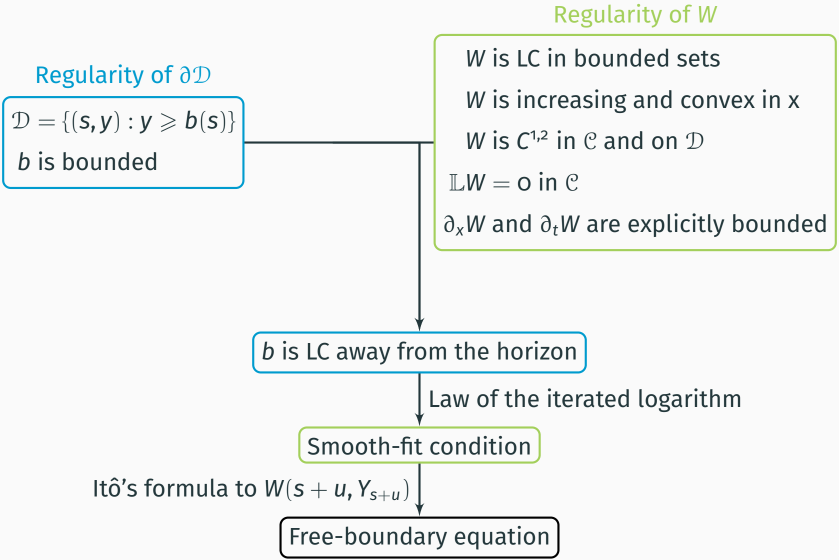
Transformed OSP: $W(s,y) := \sup_{\sigma} \mathbb{E}_{s,y}\left[G_c(s+\sigma, Y_{s+\sigma})\right]$

Original OST: $\tau^*(t,x)$

$t + \tau^*(t,x) = \gamma_T^{-1}(s + \sigma^*(s,y))$ ⟷

Transformed OST: $\sigma^*(s,y)$

Solution method for W



Solution of the transformed OSP

Theorem

- **OST:** $\sigma(s, y) = \inf\{u \in \mathbb{R}_+ : Y_{s+u} \geqslant b(s+u)\}$
- **OSB:** The unique solution, up to regularity conditions, of the integral equation

$$G_c(s, b(s)) = c_1 + c_0 c_2 - \int_s^\infty K_c(s, b(s), u, b(u))\,\mathrm{d}u$$

- **Value** function: $W(s, y) = c_1 + c_0 c_2 - \int_s^\infty K_c(s, y, u, b(u))\,\mathrm{d}u$
- **Kernel:**

$$K_c(s_1, y_1, s_2, y_2) := \left((a_1'(s_2) + c_0 a_2(s_2) + c_0 a_2'(s_2)(s_2 + y_1)\right)\overline{\Phi}\left(\frac{y_2 - y_1}{\sqrt{s_2 - s_1}}\right) + c_0 a_2'(s_2)\sqrt{s_2 - s_1}\,\phi\left(\frac{y_2 - y_1}{\sqrt{s_2 - s_1}}\right)$$

- a_1, a_2, c, c_1, and c_2 are **explicit**,
  ϕ and $\overline{\Phi}$ are the density and survival functions of a $\mathcal{N}(0, 1)$

Theorem

- **OST:** $\tau(t,x) = \inf\{u \in [0, T-t] : X_{t+u} \geqslant \mathrm{b}(t+u)\}$
- **OSB:** The unique solution, up to regularity conditions, of the integral equation

$$\mathrm{b}_{T,z}(t) = z - \int_t^T \mathrm{K}(t, \mathrm{b}_{T,z}(t), u, \mathrm{b}_{T,z}(u))\,\mathrm{d}u$$

- **Value** function: $V(t,x) = z - \int_t^T \mathrm{K}(t, x, u, \mathrm{b}(u))\,\mathrm{d}u$
- **Kernel:**

$$\mathrm{K}(t_1, x_1, t_2, x_2) = v_{t_2}(t_1)\frac{\beta_T'(t_2)}{\beta_T(t_2)}\phi\left(\frac{x_2 - m_{t_2}(t_1, x_1)}{v_{t_2}(t_1)}\right) + \overline{\Phi}\left(\frac{x_2 - m_{t_2}(t_1, x_1)}{v_{t_2}(t_1)}\right)$$
$$\times\left(\alpha'(t_2) + (m_{t_2}(t_1, x_1) - \alpha(t_2))\frac{\beta_T'(t_2)}{\beta_T(t_2)} + (z - \alpha(T))\beta_T(t_2)\gamma_T'(t_2)\right)$$

- $m_{t_2}(t_1, x_1)$ and $v_{t_2}(t_1)$ are **explicit**

Problem: The FBE is not analytically solvable

Numerical solution

- Discretization of time: $0 = t_1 < \cdots < t_N = T$
- Discretization of the FBE + Fixed-point (Picard's iteration) algorithm

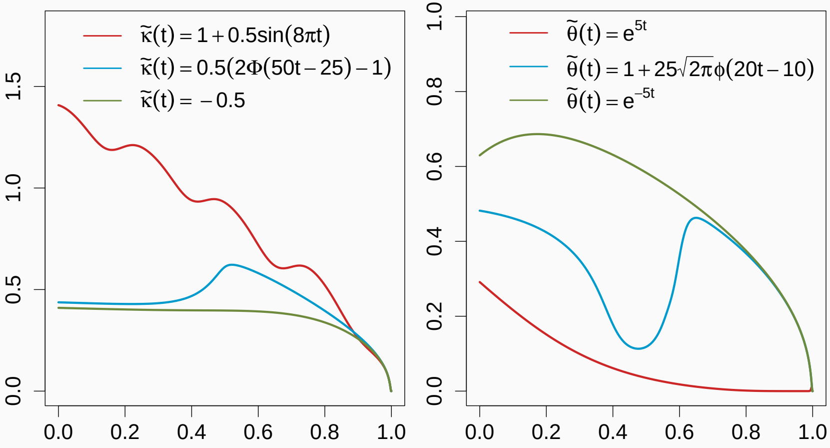
$$b^{(k)}(t_i) := z - \sum_{j=i+1}^{N-1} K\left(t_i, b^{(k-1)}(t_i), t_j, b^{(k-1)}(t_j)\right)$$

$$b^{(0)}(t) := b(T) = z, \quad \forall\, t \in [0, T]$$

- Last term of the Riemann sum excluded because the kernel explodes at maturity

Boundary computation

$$X_t = \widetilde{X}_t \,\big|\, \widetilde{X}_T = 0, \quad \mathrm{d}\widetilde{X}_t = \widetilde{\theta}(t)(\widetilde{\kappa}(t) - \widetilde{X}_t)\,\mathrm{d}t + \widetilde{\nu}(t)\,\mathrm{d}B_t, \quad \widetilde{\nu} \equiv 1$$



Varying $\widetilde{\kappa}$ with $\widetilde{\theta} \equiv 3$ — Varying $\widetilde{\theta}$ with $\widetilde{\kappa} \equiv -1$

Conclusions

Final remarks

Main contributions

- ✓ Solution of OSPs with Gauss–Markov bridges (BB, OUB, ...)
- ✓ Characterization of the OSB via Volterra integral equations
- ✓ Implementation of fixed-point algorithms to solve the FBE
- ✓ GitHub repository with the code for applicability: aguazz/OSP_GMB
- ✓ Boundedness of OSBs via comparison arguments
- ✓ Lipschitz continuity of OSBs

Future work

- Non-Gaussian Markovian bridges
- Relaxation of the OSB's smoothness to obtain the solution
- Prove the convergence of the fixed-point algorithms