Optimally Stopping a Gauss-Markov process

with random terminal value

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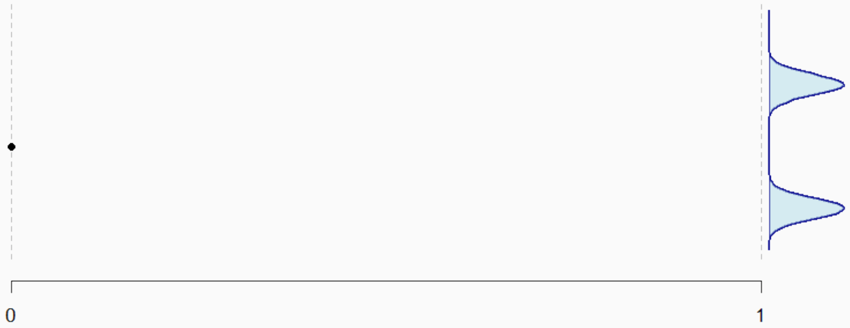
Joint work with Bernardo D'Auria (University of Padua)

Introduction

When to sell an asset if you "know" the terminal distribution?



The problem

Find the **best strategy** to sell an asset before a horizon if the trader believes in a **terminal distribution**

Randomized (Gauss-Markov) bridges

Unconditioned process

$(X_t)_{t\in[0,T]}$ in $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}})_{t\in[0,T]}, \widehat{\mathbb{P}}\right)$

Pinning point

$Z \sim \nu$ in $\left(\widetilde{\Omega}^{\nu}, \mathcal{F}^{\nu}, \widetilde{\mathbb{P}}^{\nu}\right)$

Conditioned process

$(X_t)_{t\in[0,T]}$ in $\left(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \widetilde{\mathbb{P}}^{\nu}\right)$

$$\Omega = \widehat{\Omega} \times \widetilde{\Omega}$$

$$\mathcal{F} = \widehat{\mathcal{F}} \otimes \widetilde{\mathcal{F}}$$

$$\mathcal{F}_t = \widehat{\mathcal{F}}_t \vee \sigma(Z)$$

$$\mathbb{P}^{\nu}(\cdot) = \mathbb{P} \otimes \widetilde{\mathbb{P}}^{\nu}(\cdot \mid X_T = Z)$$

Our unconditioned model choice

$(X_t)_{t\in[0,T]}$ is Markovian and Gaussian at the same time

Gauss–Markov processes[1,2,3] diffusion representation

$$\mathrm{d}X_t = (\alpha(t) + \beta(t)X_t)\,\mathrm{d}t + \gamma(t)\mathrm{d}B_t$$

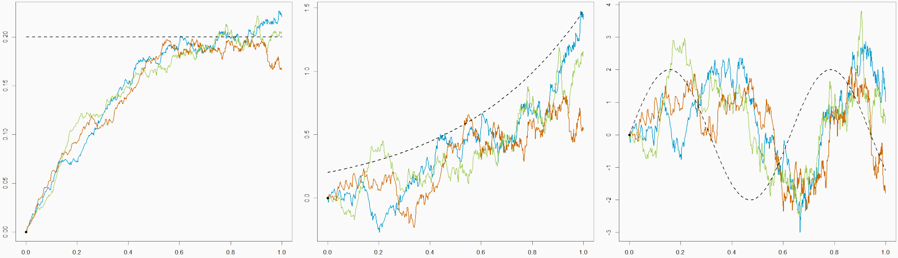


Figure: Different paths of GM processes changing the pulling level (– –) $-\beta/\alpha$

Examples: Brownian motion, Ornstein–Uhlenbeck process, Brownian bridge

[1] Mehr et al. (1965). Certain properties of Gaussian processes and their first-passage times. *J. R. Stat. Soc. Ser. B Methodol.*

[2] Borisov (1983). On a criterion for Gaussian random processes to be Markovian. *Theory Probab. Its Appl.*

[3] Buonocore et al. (2013). On some time-non-homogeneous linear diffusion processes and related bridges. *Sci. Math. Jpn.*

The Optimal Stopping Problem (OSP)

$$V(t,x) = \sup_{\tau \leqslant T-t} \mathbb{E}^{\nu}_{t,x}\left[G(X_{t+\tau})\right]$$

Dinkyn's characterization[1,2]

- V is the smallest supermartingale that dominates G
- $\tau_{\mathcal{D}}(t,x) \coloneqq \inf\left(u : X^{t,x}_u \in \mathcal{D}\right)$, with $\mathcal{D} \coloneqq \{(t,x) : V(t,x) = G(t,x)\}$, is the smallest OST

The (HJB) free-boundary problem[2]

V and ∂D (the Optimal Stopping Boundary) solves the FBP

$$\begin{aligned} \mathbb{L}V &= 0 && \text{on } \mathcal{C} \coloneqq \{x : V(t,x) > G(x)\} = \mathcal{D}^c \\ V &= G && \text{on } \mathcal{D} \\ \partial_x V &= \partial_x G && \text{on } \partial\mathcal{D} \text{ (smooth-fit condition)} \end{aligned}$$

$\mathbb{L}$ is the infinitesimal generator of (t, X_t)

[1] Dynkin (1963). The optimum choice of the instant for stopping a Markov process. *Sov. Math. Dokl.*
[2] Peskir et al. (2006). Optimal Stopping and Free-Boundary Problems. Birkhäuser

The road map

- **Problem 1:** Unconditioned Gauss–Markov processes
- **Problem 2:** Gauss–Markov bridges with deterministic pinning point ($\nu = \delta_z$)
- **Problem 3:** randomized Gauss–Markov bridges

Optimal stopping of a time-dependent Ornstein–Uhlenbeck

Azze, A., D'Auria, B., & García-Portugués, E. (2024). Optimal exercise of American options under time-dependent Ornstein–Uhlenbeck processes. Stochastics, 96(1), 921–946. https://doi.org/10.1080/17442508.2024.2325402

OSP

$$V(t,x) = \sup_{\tau \leqslant T-t} \mathbb{E}_{t,x}\left[e^{-\lambda\tau}(A - X_{t+\tau})^+\right]$$

$$\mathrm{d}X_t = \theta(t)(\alpha(t) - X_t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t$$

$$\lambda \geqslant 0, \quad A \in \mathbb{R}$$

Related problems

- ✗ Time-homogeneous[1–3]
- ✗ Infinite horizon[1]
- ✗ Smooth gain (identity,[1,2] polynomial[2])
- ✗ Partial results[4]

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[1] Taylor (1968). Optimal stopping in a Markov process. *Ann. Math. Stat.*
[2] Pedersen et al. (2000). Solving non–linear optimal stopping problems by the method of time–change. *Stoch. Anal. Appl.*
[3] Kitapbayev et al. (2017). Optimal mean-reverting spread trading: nonlinear integral equation approach. *Ann. Finance*
[4] Carr et al. (2021). Semi-analytical solutions for barrier and American options written on a time-dependent Ornstein–Uhlenbeck process. *J. Deriv.*

Solution method

Regularity of $\partial\mathcal{D}$

$$\mathcal{D} = \{(t,x) : x \leqslant b(t)\}$$

$$b(t) < A, \quad t \in [0,T)$$

$$b(T) = \min\left(A, \frac{\theta(T)\alpha(T) + \lambda A}{\theta(T) + \lambda}\right)$$

$$b(t) > -\infty, \quad t \in [0,T]$$

Regularity of V

V is Lipschitz Continuous (LC)

V is decreasing and convex in x

V is $C^{1,2}$ in $\mathcal{C}$ and on $\mathcal{D}$

$\mathbb{L}V = \lambda V$ in $\mathcal{C}$

$\partial_x V$ and $\partial_t V$ are uniformly bounded

↓

b is LC[1] on closed intervals

↓ Law of the iterated logarithm

Smooth-fit condition

↓ Itô's formula[2] to $V(s+u, Y_{s+u})$

Free-boundary equation

[1] De Angelis et al. (2019). On Lipschitz continuous optimal stopping boundaries. *SIAM J. Control Optim.*

[2] Peskir (2005a). A change-of-variable formula with local time on curves. *J. Theor. Probab.*

Solution of the OSP

- **OST:** $\tau^*(t,x) = \inf\{u \in [0, T-t] : X_{t+u} \leqslant b(t+u)\}$
- **OSB:** The unique[1] solution, up to regularity conditions, of the integral equation

$$b(t) = A - K_\lambda(A, 1, t, b(t), T, A) - \int_t^T K_\lambda(\lambda A + \theta(u)\alpha(u), \lambda + \theta(u), t, b(t), u, b(u))\,\mathrm{d}u$$

- **Value** function (pricing formula):

$$V(t,x) = \underbrace{K_\lambda(A, 1, t, x, T, A)}_{\text{European price}} + \underbrace{\int_t^T K_\lambda(\lambda A + \theta(u)\alpha(u), \lambda + \theta(u), t, x, u, b(u))\,\mathrm{d}u}_{\text{Premium}}$$

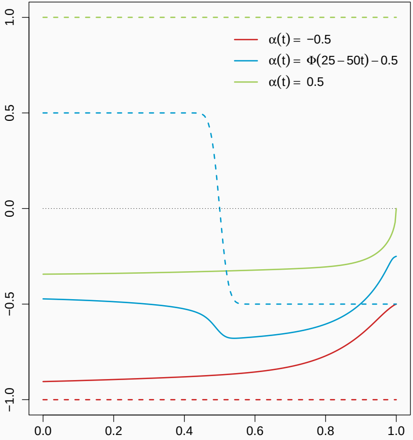
- **Kernel:** θ and γ are **explicit**

$$K_\lambda(c_1, c_2, t_1, x_1, t_2, x_2) := e^{-\lambda(t_2 - t_1)}\left\{(c_1 - c_2\theta(t_1, x_1, t_2))\Phi\left(\frac{x_2 - \theta(t_1, x_1, t_2)}{\gamma(t_1, t_2)}\right) + c_2\gamma(t_1, t_2)\phi\left(\frac{x_2 - \theta(t_1, x_1, t_2)}{\gamma(t_1, t_2)}\right)\right\}$$

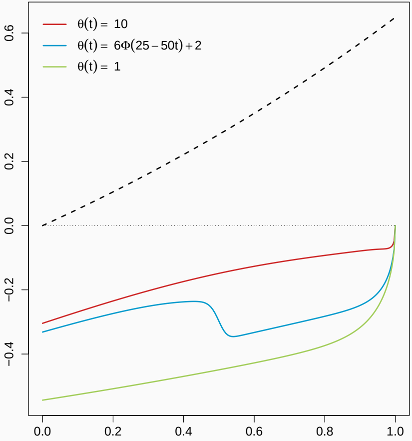
[1] Peskir (2005b). On the American option problem. *Math. Finance*

Boundary computation

$$dX_t = \theta(t)(\alpha(t) - X_t)\,dt + \sigma(t)\,dB_t, \quad \sigma \equiv 1$$



Varying α with $\theta \equiv 1$



Varying θ with $\alpha(t) = e^{0.5t}$

Optimal stopping of Gauss–Markov bridges

Azze A, D'Auria B, García-Portugués E. (2025) Optimal stopping of Gauss–Markov bridges. *Advances in Applied Probability*. 57(1):1-34. doi: 10.1017/apr.2024.21

OSP

$$V(t,x) = \sup_{\tau \leqslant T-t} \mathbb{E}_{t,x}\left[X_{t+\tau}\right], \quad X_t \text{ is a Gauss–Markov bridge}$$

Related problems

- Brownian bridge[1–3]

Gauss–Markov bridges representation

$(X_t)_{t\in[0,T]}$ admits the representation

$$\begin{cases} X_t = \alpha(t) + \beta_T(t)\left((z-\alpha(T))\gamma_T(t) + \left(B_{\gamma_T(t)} + \dfrac{x-\alpha(0)}{\beta_T(0)}\right)\right), \ t \in [0,T) \\ X_T = z \end{cases}$$

$\alpha : [0,T] \to \mathbb{R}$, $\beta_T : [0,T] \to \mathbb{R}_+$, and $\gamma_T : [0,T) \to \mathbb{R}_+$ satisfy regularity conditions

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[1] Shepp (1969). Explicit solutions to some problems of optimal stopping. *Ann. Math. Statist.*
[2] Ekström et al. (2009). Optimal stopping of a Brownian bridge. *J. Appl. Probab.*
[3] De Angelis et al. (2020). Optimal stopping for the exponential of a Brownian bridge. *J. Appl. Probab.*

Reformulation of the OSP

Time transformation: $s = \gamma_T(t)$

Space transformation: $y = \mu_T(x)$, $c = \mu_T(z)$

GMB $\longleftrightarrow$ **BM**

$$\{X_t\}_{t\in[0,T]},\ X_0 = x,\ X_T = z \quad \xleftrightarrow{\ X_t = G_c(s, Y_s)\ } \quad \{Y_s\}_{s\in\mathbb{R}_+},\ Y_0 = y$$

Original OSP $\longleftrightarrow$ **Transformed OSP**

$$V(t,x) := \sup_{\tau\leqslant T-t} \mathbb{E}_{t,x}\left[X_{t+\tau}\right] \quad \xleftrightarrow{\ V(t,x) = W(s,y)\ } \quad W(s,y) := \sup_{\sigma} \mathbb{E}_{s,y}\left[G_c(s+\sigma, Y_{s+\sigma})\right]$$

Original OST $\longleftrightarrow$ **Transformed OST**

$$\tau^*(t,x) \quad \xleftrightarrow{\ t+\tau^*(t,x) = \gamma_T^{-1}(s+\sigma^*(s,y))\ } \quad \sigma^*(s,y)$$

Solution of the original OSP

- **OST:** $\tau^*(t,x) = \inf\{u \in [0, T-t] : X_{t+u} \geqslant \mathrm{b}(t+u)\}$
- **OSB:** The unique solution, up to regularity conditions, of the integral equation

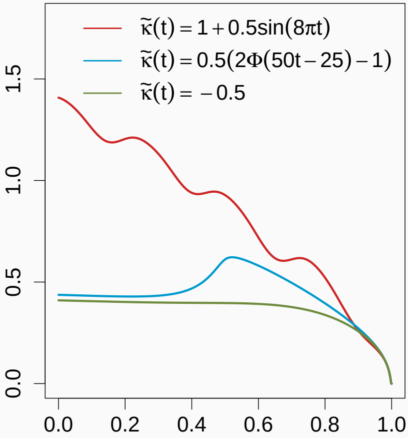
$$\mathrm{b}_{T,z}(t) = z - \int_t^T \mathrm{K}(t, \mathrm{b}_{T,z}(t), u, \mathrm{b}_{T,z}(u))\,\mathrm{d}u$$

- **Value** function: $V(t,x) = z - \int_t^T \mathrm{K}(t,x,u,\mathrm{b}(u))\,\mathrm{d}u$
- **Kernel:** $m_{t_2}(t_1,x_1)$ and $v_{t_2}(t_1)$ are **explicit**

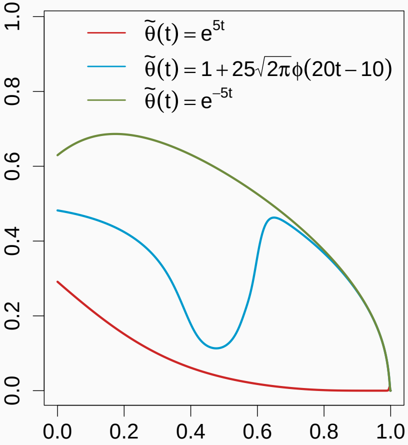
$$\mathrm{K}(t_1,x_1,t_2,x_2) = v_{t_2}(t_1)\frac{\beta_T'(t_2)}{\beta_T(t_2)}\phi\left(\frac{x_2 - m_{t_2}(t_1,x_1)}{v_{t_2}(t_1)}\right) + \overline{\Phi}\left(\frac{x_2 - m_{t_2}(t_1,x_1)}{v_{t_2}(t_1)}\right)$$
$$\times\left(\alpha'(t_2) + (m_{t_2}(t_1,x_1) - \alpha(t_2))\frac{\beta_T'(t_2)}{\beta_T(t_2)} + (z - \alpha(T))\beta_T(t_2)\gamma_T'(t_2)\right)$$

Boundary computation

$$X_t = \widetilde{X}_t \,\big|\, \widetilde{X}_T = 0, \quad \mathrm{d}\widetilde{X}_t = \widetilde{\theta}(t)(\widetilde{\kappa}(t) - \widetilde{X}_t)\,\mathrm{d}t + \widetilde{\nu}(t)\,\mathrm{d}B_t, \quad \widetilde{\nu} \equiv 1$$



Varying $\widetilde{\kappa}$ with $\widetilde{\theta} \equiv 3$



Varying $\widetilde{\theta}$ with $\widetilde{\kappa} \equiv -1$

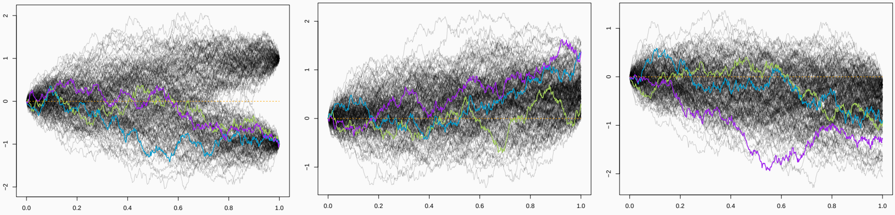
Optimal stopping of a randomized Gauss–Markov bridge

Azze A and D'Auria B. (2025) Optimal stopping of randomized Gauss–Markov bridges. ArXiv:2505.03636

OSP

$$V(t,x) = \sup_{\tau \leqslant T-t} \mathbb{E}_{t,x}\left[X_{t+\tau}\right], \quad X_t \text{ is a randomized Gauss–Markov bridge}$$

X_t is a GM process conditioned to start at $(0, x_0)$ and end at (T, Z), $Z \sim \widetilde{\nu}$



(a) $(\delta_{-1} + \delta_1)/2$ **(b)** Positive truncated N(0,0.5) **(c)** Negative truncated N(0,0.5)

Figure: Brownian bridges with different prescribed terminal densities $\widetilde{\nu}$

The transformed OSP

$$W(s,y) = \sup_{\sigma \leqslant 1-s} \mathbb{E}^{\nu}_{s,y}\left[G(s, Y_{s+\sigma})\right], \quad Y_s \text{ is a Brownian bridge with terminal density } \nu^{1,2}$$

$X_t = G(s, Y_s) = a_0(s) + a_1(s)Y_s$, for a time-change $s = s(t)$

SDE (Girsanov theorem):

$$\mathrm{d}Y_s = \mu^{\nu}(s, Y_s)\mathrm{d}s + \mathrm{d}B^{\nu}_s, \quad \mu^{\nu}(s,y) = \frac{\mathbb{E}\left[Z_{s,y}\right] - y}{1-s}, \quad \partial_y \mu^{\nu}(s,y) = \frac{1}{1-s}\left(\frac{\mathbb{V}\mathrm{ar}\left[Z_{s,y}\right]}{1-s} - 1\right),$$

$$Z_{s,y} \sim \nu_{s,y} \propto \frac{\phi(z; y, 1-s)}{\phi(z; x_0, 1)}\nu(z), \quad \phi(\cdot; \theta, \gamma^2) = N(\theta, \gamma^2)$$

Assumption: $\mathbb{V}\mathrm{ar}\left[Z_{s,y}\right]$ bounded uniformly in $(s,y) \in [0, 1-\varepsilon] \times \mathbb{R}$, for all $\varepsilon > 0$.

- The posterior $\nu_{s,y}$ (and, hence, the drift μ^{ν}) is highly non-linear in y.
- Notice the dependence with respect to the initial condition (not in the GMB case).

[1] Leung et al. (2018). Optimal timing to trade along a randomized Brownian bridge. *Int. J. Financ. Stud.*
[2] Ekström et al. (2020). Optimal stopping of a Brownian bridge with an unknown pinning point. *Stoch. Process. Their Appl.*

Disconnection of $\mathcal{D}$

$\mathcal{D}$ is not connected in general

Sufficient condition for a single boundary

If, (*i*) the **slope coefficient** of the unconditioned GM process is **non-positive**, and

(*ii*) ν is a **strongly log-concave with curvature greater than** 1 (super informative) prior, that is, $\nu(z) = f(z)\phi(z; 0, 1)$, for a log-concave function f,

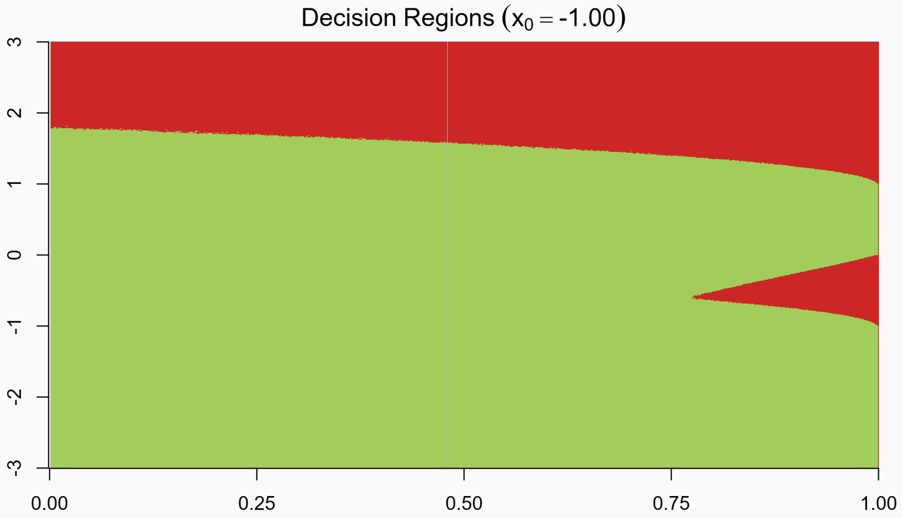
then there exists $b : [0, 1] \to \mathbb{R}$ such that

$$\mathcal{D} = \{(s, y) : y \geqslant b(s)\}.$$

Super informative priors:

- Reduce pre-conditioning variance, lighten pre-conditioning tails, unimodals

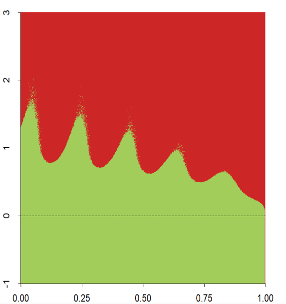
Stopping/continuation region via MC simulations. Two-points prior



Evolution of the stopping/continuation regions as the starting point x_0 varies.
Unconditioned process: standard BM; **terminal density:** $\nu = 0.5\delta_{-1} + 0.5\delta_1$

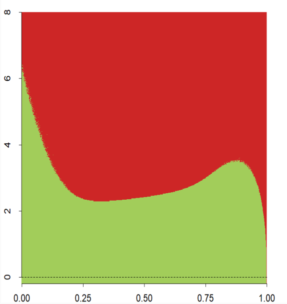
Stopping/continuation region via MC simulations. Truncated normal prior

- **The prior:** $\nu(z)$ = normal density truncated to stay positive
  $\propto \mathbb{1}(z \geqslant 0)\phi(z; 0, \mathbb{V}\mathrm{ar}^{\nu}_{0,x_0}[X_T]/2)$ ($\Longrightarrow$ ensures single boundary)
- **The unconditioned process:** $X_t = \widetilde{X}_t \,\big|\, \widetilde{X}_T \sim \nu, \quad \mathrm{d}\widetilde{X}_t = \beta(t)(\alpha(t) - \widetilde{X}_t)\,\mathrm{d}t + \zeta(t)\,\mathrm{d}B_t$
  (a) $\alpha(t) = A\sin(w\pi t), \quad A = 2, w = 10$
  (b) $\beta(t) = \theta_0 + \frac{\theta_1 - \theta_0}{2}(1 + \tanh(k(t - t_0))), \quad \theta_0 = -10, \theta_1 = 0, \quad t_0 = 0.5$
  (c) $\zeta(t) = \gamma_0 + p(t - t_0)^k, \quad \gamma_0 = 0.25, p = 4, t_0 = 0.5, k = 4$



(a) $\beta(t) \equiv 1, \quad \zeta(t) \equiv 1$

(b) $\alpha(t) \equiv 0, \quad \zeta(t) \equiv 1$



(c) $\alpha(t) \equiv 0, \quad \beta(t) \equiv -1$

Lipschitz continuity and partial derivative bounds

- Usual techniques yield, for all (s_1, y_1) and (s_2, y_2) in a compact $\mathcal{R} \subset [0, 1) \times \mathbb{R}$,

$$|W(s_1, y_1) - W(s_2, y_2)| \leqslant L(|s_1 - s_2| + |y_1 - y_2| + \mathbb{E}^{\pi}\left[|Z_{s_1,y_1} - Z_{s_2,y_2}|\right]),$$

  for $L > 0$, and where π is a joint density of the terminal points Z_{s_1,y_1}, Z_{s_2,y_2}.

- If π is selected as the Wasserstein-1 copula, that is, π^* such that

$$\mathbb{E}^{\pi^*}\left[|Z_{s_1,y_1} - Z_{s_2,y_2}|\right] = \inf_{\pi \in \Pi(\nu_{s,y_1}, \nu_{s,y_2})} \mathbb{E}^{\pi}\left[|Z_{s,y_1} - Z_{s,y_2}|\right],$$

  where $\Pi(f, g)$ is the set of all couplings whose marginal densities are f and g. Then,

$$\mathbb{E}^{\pi^*}\left[|Z_{s,y_1} - Z_{s,y_2}|\right] \leqslant L|y_1 - y_2|,$$

Free-boundary problem

The value function satisfies $W \in C^{1,2}(\mathcal{C})$ and, together with the OSB $\partial\mathcal{D}$, it solves the following FBP

$$\begin{aligned} \mathbb{L}_Y W &= 0 \quad \text{on } \mathcal{C} \\ W &= G \quad \text{on } \partial\mathcal{D} \quad \text{(instantaneous stop)} \end{aligned}$$

Note that the smooth-fit condition is missing

Comparison argument

If $\nu_1 \leqslant_{lr} \nu_2$ then $W_1 \leqslant W_2$ and, consequently, $\mathcal{C}_1 \subset \mathcal{C}_2$ and $\mathcal{D}_2 \subset \mathcal{D}_1$.

Corollary: $\partial\mathcal{D}$ is bounded (by the OSBs of GMBs) whenever ν has bounded support.

Future work

Future work

- Obtain results on the stopping/continuation geometry for rGMBs
  - Impossibility of stopping/continuation islands
  - $\partial\mathcal{D}$ as the union of graphs of functions $b_i : [t_i, 1] \to \mathbb{R}$
  - Regularity of b_i
- Non-Gaussian Markovian bridges
- Relaxation of the OSB's smoothness to obtain the solution
- Prove the convergence of the fixed-point algorithms

Thanks!