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# Optimally Stopping a Gauss-Markov process

## with random terminal value

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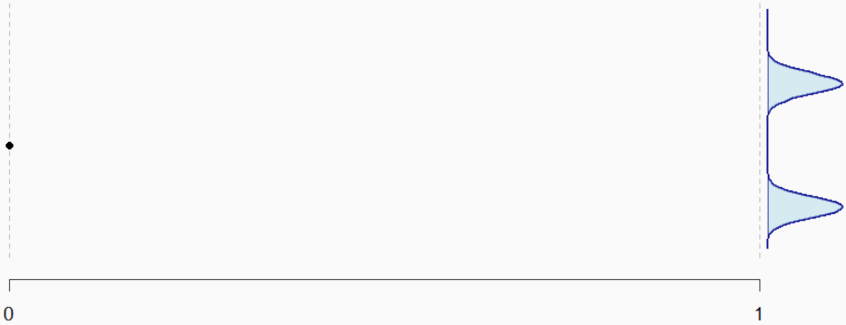
# Motivation

# When to sell an asset if you "know" the terminal distribution?




**The problem**

Find the **best strategy** to sell an asset before a horizon when the trader holds a **belief (prior) on** $X_T$.

# Randomized (Gauss-Markov) bridges

**Unconditioned process**

$(X_t)_{t\in[0,T]}$ in $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}})_{t\in[0,T]}, \widehat{\mathbb{P}}\right)$

**Pinning point**

$Z \sim \nu$ in $\left(\widetilde{\Omega}^{\nu}, \mathcal{F}^{\nu}, \widetilde{\mathbb{P}}^{\nu}\right)$

**Conditioned process**

$(X_t)_{t\in[0,T]}$ in $\left(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \widetilde{\mathbb{P}}^{\nu}\right)$

$$\Omega = \widehat{\Omega} \times \widetilde{\Omega}$$

$$\mathcal{F} = \widehat{\mathcal{F}} \otimes \widetilde{\mathcal{F}}$$

$$\mathcal{F}_t = \widehat{\mathcal{F}}_t \vee \sigma(Z)$$

$$\mathbb{P}^{\nu}(\cdot) = \mathbb{P} \otimes \widetilde{\mathbb{P}}^{\nu}(\cdot \mid X_T = Z)$$

**Our unconditioned model choice**

$(X_t)_{t\in[0,T]}$ is Markovian and Gaussian at the same time

## Gauss–Markov processes[1,2,3] diffusion representation

$$\mathrm{d}X_t = (\alpha(t) + \beta(t)X_t)\,\mathrm{d}t + \gamma(t)\mathrm{d}B_t$$

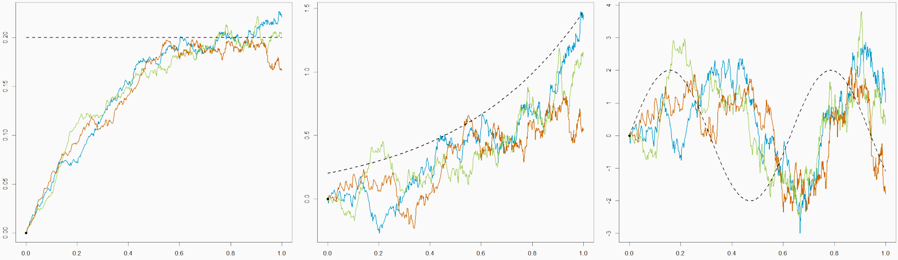


**Figure:** Different paths of GM processes changing the pulling level (— —) $-\beta/\alpha$

**Examples:** Brownian motion, Ornstein–Uhlenbeck process, Brownian bridge

[1] Mehr et al. (1965). Certain properties of Gaussian processes and their first-passage times. *J. R. Stat. Soc. Ser. B Methodol.*
[2] Borisov (1983). On a criterion for Gaussian random processes to be Markovian. *Theory Probab. Its Appl.*
[3] Buonocore et al. (2013). On some time-non-homogeneous linear diffusion processes and related bridges. *Sci. Math. Jpn.*

**The Optimal Stopping Problem (OSP)**

$$V(t,x) = \sup_{\tau \leqslant T-t} \mathbb{E}^{\nu}_{t,x}\left[G(X_{t+\tau})\right]$$

**Dinkyn's characterization**[1,2]

- $V$ is the smallest supermartingale that dominates $G$
- $\tau_{\mathcal{D}}(t,x) \coloneqq \inf\left(u : X_u^{t,x} \in \mathcal{D}\right)$, with $\mathcal{D} \coloneqq \{(t,x) : V(t,x) = G(t,x)\}$, is the smallest OST

**History:**

- **Problem 1:** Time-dependent Ornstein-Uhlenebeck process ($\nu = \mathcal{N}$)
- **Problem 2:** Gauss–Markov bridges with deterministic pinning point ($\nu = \delta_z$)
- **Problem 3:** randomized Gauss–Markov bridges

[1] Dynkin (1963). The optimum choice of the instant for stopping a Markov process. *Sov. Math. Dokl.*
[2] Peskir et al. (2006). Optimal Stopping and Free-Boundary Problems. Birkhäuser

# Optimal stopping of a time-dependent Ornstein–Uhlenbeck

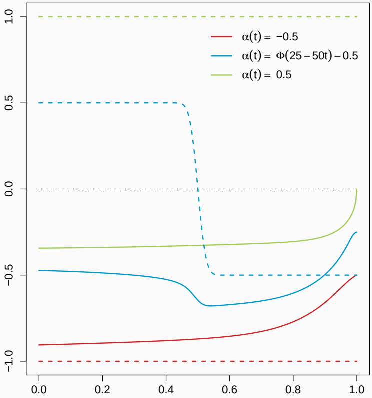
Azze, A., D’Auria, B., & García-Portugués, E. (2024). Optimal exercise of American options under time-dependent Ornstein–Uhlenbeck processes. Stochastics, 96(1), 921–946. https://doi.org/10.1080/17442508.2024.2325402

# The OSP

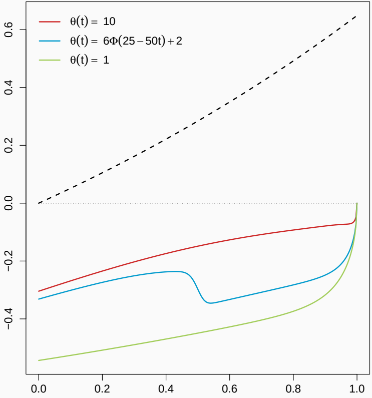
## OSP

$$V(t,x) = \sup_{\tau \leqslant T-t} \mathbb{E}_{t,x}\left[e^{-\lambda\tau}(A - X_{t+\tau})^{+}\right],$$

$$\mathrm{d}X_t = \theta(t)(\alpha(t) - X_t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t, \quad \theta > 0,\ \lambda \geqslant 0,\ A \in \mathbb{R}$$




Varying $\alpha$ with $\theta \equiv 1$




Varying $\theta$ with $\alpha(t) = e^{0.5t}$

# Solution method

**Regularity of $\partial\mathcal{D}$**

$$\mathcal{D} = \{(t, x) : x \leqslant b(t)\}$$

$$b(t) < A, \quad t \in [0, T)$$

$$b(T) = \min\left(A, \frac{\theta(T)\alpha(T) + \lambda A}{\theta(T) + \lambda}\right)$$

$$b(t) > -\infty, \quad t \in [0, T]$$

**Regularity of $V$**

$V$ is Lipschitz Continuous (LC)

$V$ is decreasing and convex in x

$V$ is $C^{1,2}$ in $\mathcal{C}$ and on $\mathcal{D}$

$\mathbb{L}V = \lambda V$ in $\mathcal{C}$

$\partial_x V$ and $\partial_t V$ are uniformly bounded

↓

**$b$ is LC[1] on closed intervals**

↓ Law of the iterated logarithm

**Smooth-fit condition**

↓ Itô's formula[2] to $V(s+u, Y_{s+u})$

**Free-boundary equation**

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[1] De Angelis et al. (2019). On Lipschitz continuous optimal stopping boundaries. *SIAM J. Control Optim.*

[2] Peskir (2005a). A change-of-variable formula with local time on curves. *J. Theor. Probab.*

## Solution of the OSP

- **OST:** $\tau^*(t,x) = \inf\{u \in [0, T-t] : X_{t+u} \leqslant b(t+u)\}$
- **OSB:** The unique[1] solution, up to regularity conditions, of the integral equation

$$b(t) = A - K_\lambda(A, 1, t, b(t), T, A) - \int_t^T K_\lambda(\lambda A + \theta(u)\alpha(u), \lambda + \theta(u), t, b(t), u, b(u))\,\mathrm{d}u$$

- **Value** function (pricing formula):

$$V(t,x) = \underbrace{K_\lambda(A, 1, t, x, T, A)}_{\text{European price}} + \underbrace{\int_t^T K_\lambda(\lambda A + \theta(u)\alpha(u), \lambda + \theta(u), t, x, u, b(u))\,\mathrm{d}u}_{\text{Premium}}$$

- **Kernel:** $\theta$ and $\gamma$ are **explicit**

$$K_\lambda(c_1, c_2, t_1, x_1, t_2, x_2) := e^{-\lambda(t_2 - t_1)}\left\{(c_1 - c_2\theta(t_1, x_1, t_2))\Phi\left(\frac{x_2 - \theta(t_1, x_1, t_2)}{\gamma(t_1, t_2)}\right) + c_2\gamma(t_1, t_2)\phi\left(\frac{x_2 - \theta(t_1, x_1, t_2)}{\gamma(t_1, t_2)}\right)\right\}$$

[1] Peskir (2005b). On the American option problem. *Math. Finance*

## Optimal stopping of Gauss–Markov bridges

Azze A, D'Auria B, García-Portugués E. (2025) Optimal stopping of Gauss–Markov bridges. *Advances in Applied Probability*. 57(1):1-34. doi: 10.1017/apr.2024.21

### OSP

$$V(t,x) = \sup_{\tau \leqslant T-t} \mathbb{E}_{t,x}\left[X_{t+\tau}\right], \quad X_t \text{ is a Gauss–Markov bridge}$$

### Related problems

- Brownian bridge[1–3]

### Gauss–Markov bridges representation

$(X_t)_{t\in[0,T]}$ admits the representation

$$\begin{cases} X_t = \alpha(t) + \beta_T(t)\left((z-\alpha(T))\gamma_T(t) + \left(B_{\gamma_T(t)} + \dfrac{x-\alpha(0)}{\beta_T(0)}\right)\right), \ t \in [0,T) \\ X_T = z \end{cases}$$

$\alpha : [0,T] \to \mathbb{R}$, $\beta_T : [0,T] \to \mathbb{R}_+$, and $\gamma_T : [0,T) \to \mathbb{R}_+$ satisfy regularity conditions

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[1] Shepp (1969). Explicit solutions to some problems of optimal stopping. *Ann. Math. Statist.*

[2] Ekström et al. (2009). Optimal stopping of a Brownian bridge. *J. Appl. Probab.*

[3] De Angelis et al. (2020). Optimal stopping for the exponential of a Brownian bridge. *J. Appl. Probab.*

# Reformulation of the OSP

Time transformation: $s = \gamma_T(t)$

Space transformation: $y = \mu_T(x)$, $c = \mu_T(z)$

**GMB**: $\{X_t\}_{t\in[0,T]},\ X_0 = x,\ X_T = z$

$\longleftrightarrow$ $X_t = G_c(s, Y_s)$ $\longleftrightarrow$

**BM**: $\{Y_s\}_{s\in\mathbb{R}_+},\ Y_0 = y$

**Original OSP**: $V(t,x) := \sup_{\tau\leqslant T-t} \mathbb{E}_{t,x}\left[X_{t+\tau}\right]$

$\longleftrightarrow$ $V(t,x) = W(s,y)$ $\longleftrightarrow$

**Transformed OSP**: $W(s,y) := \sup_{\sigma} \mathbb{E}_{s,y}\left[G_c(s+\sigma, Y_{s+\sigma})\right]$

**Original OST**: $\tau^*(t,x)$

$\longleftrightarrow$ $t + \tau^*(t,x) = \gamma_T^{-1}(s + \sigma^*(s,y))$ $\longleftrightarrow$

**Transformed OST**: $\sigma^*(s,y)$

# Solution of the original OSP

- **OST:** $\tau^*(t,x) = \inf\{u \in [0, T-t] : X_{t+u} \geqslant \mathrm{b}(t+u)\}$
- **OSB:** The unique solution, up to regularity conditions, of the integral equation

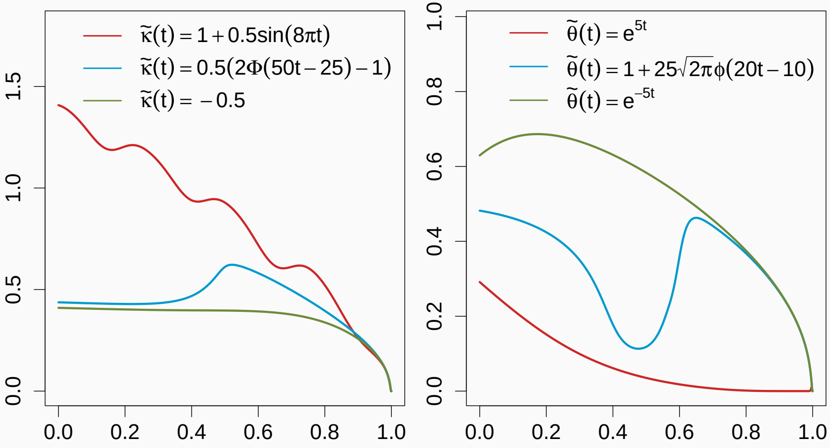
$$\mathrm{b}_{T,z}(t) = z - \int_t^T \mathrm{K}(t, \mathrm{b}_{T,z}(t), u, \mathrm{b}_{T,z}(u))\,\mathrm{d}u$$

- **Value** function: $V(t,x) = z - \int_t^T \mathrm{K}(t,x,u,\mathrm{b}(u))\,\mathrm{d}u$
- **Kernel:** $m_{t_2}(t_1,x_1)$ and $v_{t_2}(t_1)$ are **explicit**

$$\begin{aligned}\mathrm{K}(t_1,x_1,t_2,x_2) = {}& v_{t_2}(t_1)\frac{\beta_T'(t_2)}{\beta_T(t_2)}\phi\left(\frac{x_2 - m_{t_2}(t_1,x_1)}{v_{t_2}(t_1)}\right) + \overline{\Phi}\left(\frac{x_2 - m_{t_2}(t_1,x_1)}{v_{t_2}(t_1)}\right)\\ &\times\left(\alpha'(t_2) + (m_{t_2}(t_1,x_1) - \alpha(t_2))\frac{\beta_T'(t_2)}{\beta_T(t_2)} + (z - \alpha(T))\beta_T(t_2)\gamma_T'(t_2)\right)\end{aligned}$$

# Boundary computation

$$X_t = \widetilde{X}_t \,\big|\, \widetilde{X}_T = 0, \quad \mathrm{d}\widetilde{X}_t = \widetilde{\theta}(t)(\widetilde{\kappa}(t) - \widetilde{X}_t)\,\mathrm{d}t + \widetilde{\nu}(t)\,\mathrm{d}B_t, \quad \widetilde{\nu} \equiv 1$$




Varying $\widetilde{\kappa}$ with $\widetilde{\theta} \equiv 3$ Varying $\widetilde{\theta}$ with $\widetilde{\kappa} \equiv -1$

## Optimal stopping of a randomized Gauss–Markov bridge

Azze A and D'Auria B. (2025) Optimal stopping of randomized Gauss–Markov bridges. ArXiv:2505.03636

## Original OSP

$$V(t,x) = \sup_{\tau \leqslant T-t} \mathbb{E}_{t,x}\left[X_{t+\tau}\right], \quad X_t \text{ is a randomized Gauss–Markov bridge}$$

$X_t$ is a GM process conditioned to start at $(0, x_0)$ and end at $(T, Z)$, $Z \sim \widetilde{\nu}$

### Related results

- Brownian motion ($\nu = \mathcal{N}$)
- Ornstein-Uhlenbeck[1–5] ($\nu = \mathcal{N}$)
- Brownian bridge[6–9] ($\nu = \delta_z$)
- Gauss–Markov bridge[10] ($\nu = \delta_z$)
- Randomized Brownain bridge[11,12]

[1] Taylor (1968). Optimal stopping in a Markov process. *Ann. Math. Stat.*
[2] Pedersen et al. (2000). Solving non–linear optimal stopping problems by the method of time–change. *Stoch. Anal. Appl.*
[3] Kitapbayev et al. (2017). Optimal mean-reverting spread trading: nonlinear integral equation approach. *Ann. Finance*
[4] Carr et al. (2021). Semi-analytical solutions for barrier and American options written on a time-dependent Ornstein–Uhlenbeck process. *J. Deriv.*
[5] Azze et al. (2024a). Optimal exercise of American options under time-dependent Ornstein–Uhlenbeck processes. *Stochastics*
[6] Ekström et al. (2009). Optimal stopping of a Brownian bridge. *J. Appl. Probab.*
[7] De Angelis et al. (2020). Optimal stopping for the exponential of a Brownian bridge. *J. Appl. Probab.*
[8] D'Auria et al. (2020a). A class of Itô diffusions with known terminal value and specified optimal barrier. *Mathematics*
[9] D'Auria et al. (2020b). Discounted optimal stopping of a Brownian bridge, with application to American options under pinning. *Mathematics*
[10] Azze et al. (2024b). Optimal stopping of Gauss–Markov bridges. *Adv. Appl. Probab.*
[11] Leung et al. (2018). Optimal timing to trade along a randomized Brownian bridge. *Int. J. Financ. Stud.*
[12] Ekström et al. (2020). Optimal stopping of a Brownian bridge with an unknown pinning point. *Stoch. Process. Their Appl.*

# The transformed OSP of a rBB

## The transformed OSP

$$W(s,y) = \sup_{\sigma \leqslant 1-s} \mathbb{E}^{\nu}_{s,y}\left[G(s, Y_{s+\sigma})\right],$$

$X_t = G(s, Y_s) = a_0(s) + a_1(s)Y_s$ is a Brownian bridge with terminal density $\nu$[1,2]

**SDE via Girsanov's Theorem:**

$$\mathrm{d}Y_s = \mu^{\nu}(s, Y_s)\mathrm{d}s + \mathrm{d}B^{\nu}_s, \quad \mu^{\nu}(s,y) = \frac{\mathbb{E}\left[Z_{s,y}\right] - y}{1-s},$$

with the posterior

$$Z_{s,y} \sim \nu_{s,y} \propto \frac{\phi(z; y, 1-s)}{\phi(z; x_0, 1)}\nu(z), \quad \phi(\cdot;\theta,\gamma^2) = N(\theta,\gamma^2)$$

**Assumption:** $\mathbb{V}\mathrm{ar}\left[Z_{s,y}\right]$ bounded uniformly in $(s,y) \in [0, 1-\varepsilon] \times \mathbb{R}$, for all $\varepsilon > 0$.

- Drift's spacial-derivative may explode at the horizon: $\partial_y \mu^{\nu}(s,y) = \frac{1}{1-s}\left(\frac{\mathbb{V}\mathrm{ar}\left[Z_{s,y}\right]}{1-s} - 1\right)$
- The posterior $\nu_{s,y}$ (and, hence, the drift $\mu^{\nu}$) is highly non-linear in $y$.
- Notice the dependence with respect to the initial condition (not in the GMB case).

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[1] Leung et al. (2018). Optimal timing to trade along a randomized Brownian bridge. *Int. J. Financ. Stud.*
[2] Ekström et al. (2020). Optimal stopping of a Brownian bridge with an unknown pinning point. *Stoch. Process. Their Appl.*

**First-entry optimality**

The stopping time

$$\sigma^*(s,y) := \inf\{u \in [0,1-s] : \ G(s+u, Y_{s+u}) \in \mathcal{D}\}$$

is optimal, and $W$ is the Snell envelope of $G(s,Y_s)$ under the rBB.

*Proof.*

- Tower property and BB[1] maximal bounds ensure integrability:

$$\mathbb{E}^{\nu}_{s,y}\left[\sup_{u \leqslant 1-s} |Y_{s+u}|\right] = \mathbb{E}^{\nu}_{s,y}\left[\mathbb{E}^{\nu}_{s,y}\left[\sup_{u \leqslant 1-s} |Y_{s+u}| \;\middle|\; Z_{s,y}\right]\right] \leqslant \sqrt{2\pi}\ln(2) + |y| + \mathbb{E}^{\nu}\left[|Z_{s,y}|\right] < \infty.$$

- $G$ is linear in $y$.

[1] recurrent technique

## Lipschitz continuity and partial derivative bounds

- For all $(s_1, y_1)$ and $(s_2, y_2)$ in a compact $\mathcal{R} \subset [0, 1) \times \mathbb{R}$,

$$|W(s_1, y_1) - W(s_2, y_2)| \leqslant L(|s_1 - s_2| + |y_1 - y_2| + \mathbb{E}^{\pi}[|Z_{s_1,y_1} - Z_{s_2,y_2}|]),$$

  for $L > 0$, and where $\pi$ is a joint density of the terminal points $Z_{s_1,y_1}$, $Z_{s_2,y_2}$.
- If $\pi$ is selected as the Wasserstein-1 copula, that is, $\pi^*$ such that

$$\mathbb{E}^{\pi^*}[|Z_{s_1,y_1} - Z_{s_2,y_2}|] = \inf_{\pi \in \Pi(\nu_{s,y_1}, \nu_{s,y_2})} \mathbb{E}^{\pi}[|Z_{s,y_1} - Z_{s,y_2}|],$$

  where $\Pi(f, g)$ is the set of all couplings whose marginal densities are $f$ and $g$. Then,

$$\mathbb{E}^{\pi^*}[|Z_{s,y_1} - Z_{s,y_2}|] \leqslant L|y_1 - y_2|,$$

**Free-boundary problem**

The value function satisfies $W \in C^{1,2}(\mathcal{C})$ and, together with the OSB $\partial\mathcal{D}$, it solves the following FBP

$$\begin{aligned} \mathbb{L}_\gamma W &= 0 \quad \text{on } \mathcal{C} \\ W &= G \quad \text{on } \partial\mathcal{D} \quad \text{(instantaneous stop)} \end{aligned}$$

Note that the smooth-fit condition is missing

# Comparison argument via likelihood-ratio order of priors

**Comparison argument**

If $\nu_1 \leqslant_{lr} \nu_2$ then $W_1 \leqslant W_2$ and, consequently, $\mathcal{C}_1 \subset \mathcal{C}_2$ and $\mathcal{D}_2 \subset \mathcal{D}_1$.

**Corollaries:**

- If $\sup\{\mathrm{supp}(\nu_1)\} \leqslant \inf\{\mathrm{supp}(\nu_2)\}$, then $\mathcal{D}_2 \subset \mathcal{D}_1$
- If $\nu_1 \sim \mathcal{N}(\theta_1, \gamma^2)$ and $\nu_2 \sim \mathcal{N}(\theta_2, \gamma^2)$, then $\mathcal{D}_2 \subset \mathcal{D}_1$.
- If $\nu$ has bounded support, then $\partial\mathcal{D}$ is bounded between the OSBs of GMBs deterministically pinned to the larger and lowest points of $\mathrm{supp}(\nu)$.

# Proof of LR comparison argument

**Step 1. Posterior LR order.** Let $\nu_{i;s,y}$ be the posterior of $Z$ given $(s, Y_s = y)$ under prior $\nu_i$. Recall that

$$\nu_{i,s,y} \propto \phi(z; y, 1-s)\nu_i(z).$$

Hence,

$$\frac{\nu_{1,s,y}}{\nu_{2,s,y}} \propto \frac{\nu_1(z)}{\nu_2(z)},$$

which implies $\nu_{1,s,y} \leqslant_{lr} \nu_{2,s,y}$.

**Step 2. Means and drifts.** By LR order[1], $\mathbb{E}_{\nu_{1;s,y}}[Z] \leqslant \mathbb{E}_{\nu_{2;s,y}}[Z]$. Using the drift identity for the rBB,

$$\mu^{\nu_i}(s,y) = \frac{\mathbb{E}_{\nu_{i;s,y}}[Z] - y}{1-s},$$

we obtain $\mu^{\nu_1}(s,y) \leqslant \mu^{\nu_2}(s,y)$ for all $(s,y)$.

**Step 3. Pathwise comparison.** Under a coupling with the same Brownian motion and independent pinning points, the rBBs satisfy $Y^{(1)}_{s+u} \leqslant Y^{(2)}_{s+u}$ a.s. for all $u$.

**Step 4. Values and regions ordering.** Since $G(s,y) = a_0(s) + a_1(s)y$ with $a_1(s) > 0$,

$$\mathbb{E}\Big[G(s+\sigma, Y^{(1)}_{s+\sigma})\Big] \leqslant \mathbb{E}\Big[G(s+\sigma, Y^{(2)}_{s+\sigma})\Big] \quad \forall\, \sigma \leqslant 1-s.$$

Taking $\sup_\sigma$ yields $\boldsymbol{W_1} \leqslant \boldsymbol{W_2}$, hence $\mathcal{D}_2 \subset \mathcal{D}_1$ and $\mathcal{C}_1 \subset \mathcal{C}_2$.

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[1] Shaked et al., eds. (2007). Stochastic Orders. 1st. Springer (Thm 1.C.1)

[2] Peng et al. (2006). Necessary and sufficient condition for comparison theorem of 1-dimensional stochastic differential equations. *Stoch. Process. Their Appl.* (Cor. 3.1)

### Disconnection of $\mathcal{D}$

$\mathcal{D}$ need not be connected for arbitrary priors (e.g., two-atoms masses).

### Sufficient condition for a single boundary

**Proposition.** Let $W$ be the rBB value function and $\mathcal{D} = \{W = G\}$ the stopping set.

(i) If:

- $(i.1)$ $y \mapsto \mu^{\gamma}(s, y)$ is decreasing for all $s \in [0, 1)$ (equivalently $\mathrm{Var}[Z_{s,y}] \leqslant 1 - s$), and
- $(i.2)$ $a_1$ is decreasing (equiv., **negative slope coefficient of the unconditioned GM process**)

Then $\exists\, b : [0, 1] \to [-\infty, \infty]$ with

$$\mathcal{D} = \{(s, y) : \ y \geqslant b(s)\}.$$

(ii) Analogous conditions to obtain $\mathcal{D} = \{(s, y) : \ y \leqslant b(s)\}$.

**Super informative priors:**

- Reduce pre-conditioning variance

# Proof of case (i): upper single boundary

**Step 0: Variance–drift link.** Recall that $\partial_y \mu^\nu(s,y) = \frac{1}{1-s}\left(\frac{\mathrm{Var}[Z_{s,y}]}{1-s} - 1\right)$, meaning that $\mathrm{Var}[Z_{s,y}] \geqslant 1-s \iff y \mapsto \mu^\nu(s,y)$ is non-increasing.

**Step 1: Dynkin/Itô for $G$.** For any stopping time $\sigma \leqslant 1-s$,

$$W(s,y) - G(s,y) = \sup_{\sigma \leqslant 1-s} \mathbb{E}\left[\int_0^\sigma (\mathbb{L}_Y G)(s+u, Y_u^{s,y})\, du\right],$$

where (since $G(s,y) = a_0(s) + a_1(s)y$)

$$(\mathbb{L}_Y G)(s,y) = \partial_s G + \mu^\nu(s,y)\, \partial_y G = a_0'(s) + a_1'(s)\, y + a_1(s)\, \mu^\nu(s,y).$$

**Step 2: Monotonicity of the integrand.** If $a_1'(s) \leqslant 0$ and $y \mapsto \mu^\nu(s,y)$ is non-increasing, then $y \mapsto (\mathbb{L}_Y G)(s,y)$ is non-increasing for every $s$.

**Step 3: Order preservation of rBB.** For $y_1 \leqslant y_2$, $Y_u^{s,y_1} \leqslant Y_u^{s,y_2}$ a.s. for all $u \in [0, 1-s]$.

**Step 4: Monotonicity of $W - G$.** Combining Steps 1–3, $y \mapsto (W-G)(s,y)$ is non-increasing. Hence, if $(s,y_1) \in \mathcal{D}$, then $(W-G)(s,y_2) \leqslant 0$ for every $y_2 \geqslant y_1$, so $(s,y_2) \in \mathcal{D}$. Therefore $\mathcal{D} = \{y \geqslant b(s)\}$.

**Corollary of the single-boundary proposition**

If, (*i*) the **slope coefficient** of the unconditioned GM process is **non-positive**, and

(*ii*) $\nu$ is a **strongly log-concave with curvature greater than** 1 (SLC$_1$) prior, that is, $\nu(z) = f(z)\phi(z; 0, 1)$, for a log-concave function $f$,

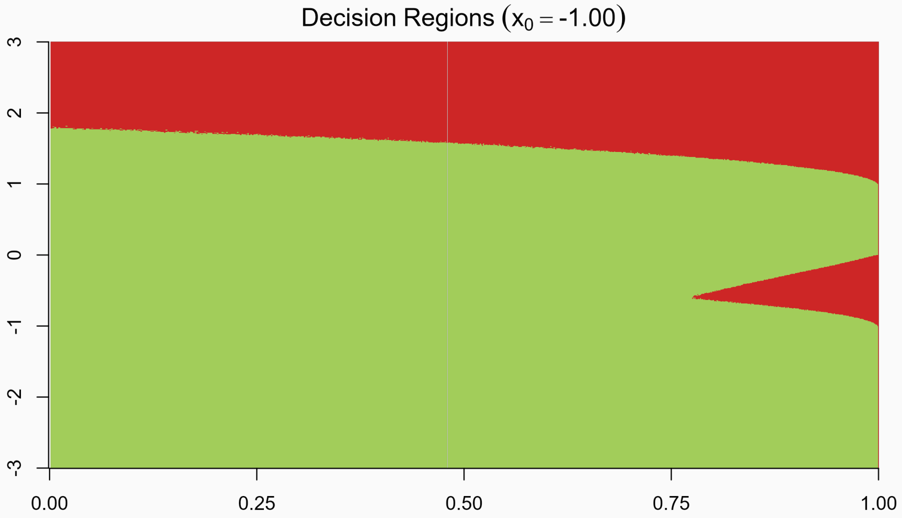
then there exists $b : [0, 1] \to [-\infty, \infty]$ such that

$$\mathcal{D} = \{(s, y) : y \geqslant b(s)\}.$$

**SLC$_1$ priors:**

- Reduce pre-conditioning variance, lighten pre-conditioning tails, unimodals

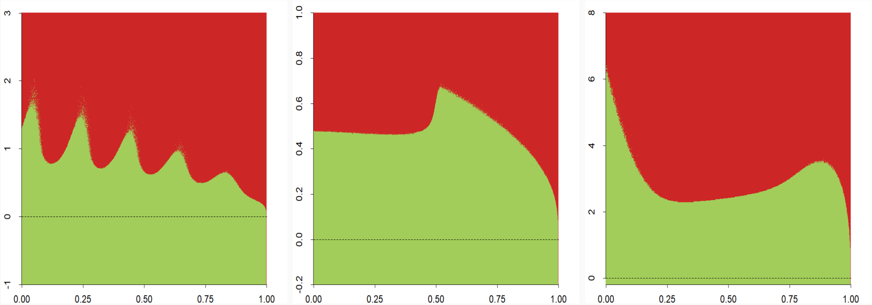
# Stopping/continuation region via MC simulations. Two-points prior




Evolution of the stopping/continuation regions as the starting point $x_0$ varies.
**Unconditioned process:** standard BM; **terminal density:** $\nu = 0.5\delta_{-1} + 0.5\delta_1$

## Stopping/continuation region via MC simulations. Truncated normal prior

- **The prior:** $\nu(z)$ = normal density truncated to stay positive
  $\propto \mathbb{1}(z \geqslant 0)\phi(z; 0, \mathbb{V}\mathrm{ar}^{\nu}_{0,x_0}[X_T]/2)$ ($\Longrightarrow$ ensures single boundary)
- **The unconditioned process:** $X_t = \widetilde{X}_t \,\big|\, \widetilde{X}_T \sim \nu, \quad \mathrm{d}\widetilde{X}_t = \beta(t)(\alpha(t) - \widetilde{X}_t)\,\mathrm{d}t + \zeta(t)\,\mathrm{d}B_t$
  (a) $\alpha(t) = A\sin(w\pi t), \quad A = 2, w = 10$
  (b) $\beta(t) = \theta_0 + \frac{\theta_1 - \theta_0}{2}(1 + \tanh(k(t - t_0))), \quad \theta_0 = -10, \theta_1 = 0, \quad t_0 = 0.5$
  (c) $\zeta(t) = \gamma_0 + p(t - t_0)^k, \quad \gamma_0 = 0.25, p = 4, t_0 = 0.5, k = 4$




**(a)** $\beta(t) \equiv 1, \quad \zeta(t) \equiv 1$ **(b)** $\alpha(t) \equiv 0, \quad \zeta(t) \equiv 1$ **(c)** $\alpha(t) \equiv 0, \quad \beta(t) \equiv -1$

# Future work

## Future work

- Obtain results on the stopping/continuation geometry for rGMBs
  - Impossibility of stopping/continuation islands
  - $\partial\mathcal{D}$ as the union of graphs of functions $b_i : [t_i, 1] \to \mathbb{R}$
  - Regularity of $b_i$
- Non-Gaussian Markovian bridges
- Relaxation of the OSB's smoothness to obtain the solution
- Prove the convergence of the fixed-point algorithms

Thanks!