Ergodic singular control for **ambiguous** compound-Poisson **jump-diffusion** processes

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Motivation

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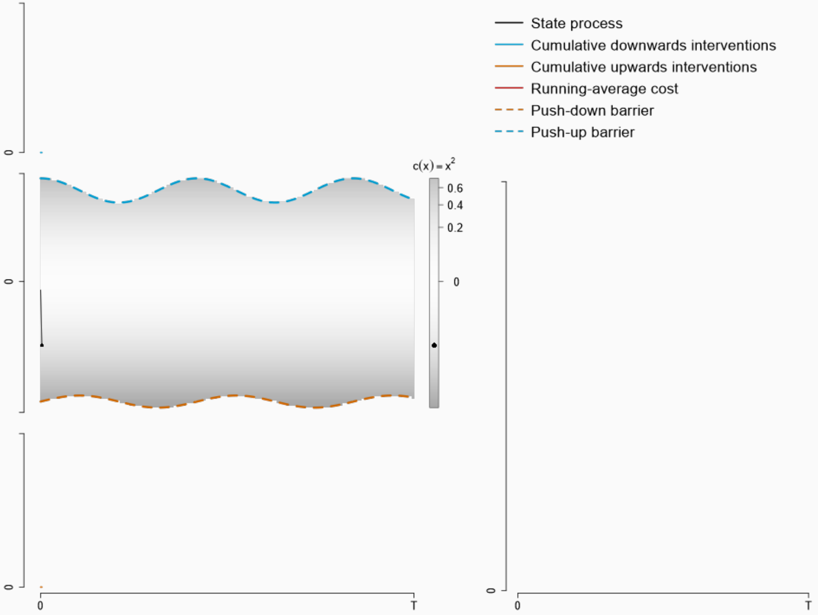
- **Goal:** Minimize **average long-run** (ergodic) cost.
  $\implies$ Ergodic singular control!

Visualizing the cash management problem



The Ergodic Singular Problem

Ergodic singular control problem

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- $X_t^{U,D}$: controlled process

$$\mathrm{d}X_t^{U,D} = b(X_{t-}^{U,D})\,\mathrm{d}t + \sigma\,\mathrm{d}B_t + \mathrm{d}\widetilde{Z}_t + \mathrm{d}U_t - \mathrm{d}D_t,$$

for:

  - $b(\cdot)$: Lipschitz-continuous drift function;
  - σ: volatility;
  - B_t: $\mathbb{P}^0$-standard Brownian motion;
  - $\widetilde{Z}_t$: $\mathbb{P}^0$-compensated compound-Poisson process (independent from B_t) with intensity $r > 0$ and i.i.d. jumps $\{Y_n\}_{n=1}^{\infty}$ with jump-size distribution F.

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- γ: **ergodic value.**
- $\mathbb{E}_x^0$: expectation w.r.t. the **benchmark measure** $\mathbb{P}_x^0$, indicating $\mathbb{P}_x^0(X_0 = x) = 1$.

Model Ambiguity

AMBIGUITY[1]

We do not trust the probability measure!

To capture misspecification (in the drift and intensity) we consider the family $\{\mathbb{P}^{\kappa,\lambda}\}_{(\kappa,\lambda)\in\Lambda}$ whose RN densities w.r.t. $\mathbb{P}^0$ are given by

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[1] Knight (1921). Risk, Uncertainty and Profit. Houghton Mifflin Company

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- $\kappa_t = \kappa(X_t)$ and $\lambda_t = \lambda(X_t)$: state-dependent drift and intensity distortions.
- $\kappa(\cdot)$ and $\lambda(\cdot)$: drift and intensity distortion functions
- Λ: set of admissible distortions:

$$|\kappa(x)| \leqslant \delta, \quad (1-\varepsilon)r \leqslant \lambda(x) \leqslant r(1+\varepsilon),$$

for $\delta > 0$ and $0 \leqslant \varepsilon < 1$.

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$$|\kappa(x)| \leqslant \delta, \quad (1-\varepsilon)r \leqslant \lambda(x) \leqslant r(1+\varepsilon),$$

  for $\delta > 0$ and $0 \leqslant \varepsilon < 1$.
- Under $\mathbb{P}^{\kappa,\lambda}$:

$$dX_t^{U,D} = (b(X_{t-}^{U,D}) + \sigma\kappa(X_{t-}^{U,D}) + (\lambda(X_{t-}^{U,D}) - r)\,\mathbb{E}[Y])\,dt + \sigma\, dB_t^{\kappa} + d\widetilde{Z}_t^{\lambda} + dU_t - dD_t,$$

  for:
  - B_t^{κ}: $\mathbb{P}^{\kappa,\lambda}$-standard Brownian motion;
  - $\widetilde{Z}_t^{\lambda}$: $\mathbb{P}^{\kappa,\lambda}$-compensated "compound-Poisson process" (independent from B_t^{κ}) with space-dependent intensity $\lambda(X_{t-}^{U,D})$ and i.i.d. jumps $\{Y_n\}_{n=1}^{\infty}$ with distr. F.

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Robust Ergodic singular control problem

$$\gamma = \inf_{(U,D)\in\mathcal{A}} \sup_{(\kappa,\lambda)\in\Lambda} J_x(U,D,\kappa,\lambda)$$

- **Robustness:** the controller is ambiguity averse: worst-case scenario.
- **Ergodic cost:** $J_x(U,D,\kappa,\lambda) := \lim_{T\to\infty} \frac{1}{T}\,\mathbb{E}_x^{\kappa,\lambda}\left[\int_0^T c(X_t^{U,D})\,\mathrm{d}t + c_U U_T + c_D D_T\right]$.
- **Solving the problem means finding:**
  - (U^*, D^*): optimal controls;
  - (κ^*, λ^*) optimal ambiguity functions,

  such that there is a saddle point at $(U^*, D^*, \kappa^*, \lambda^*)$:

$$\gamma = \inf_{(U,D)\in\mathcal{A}} \sup_{(\kappa,\lambda)\in\Lambda} J_x(U,D,\kappa,\lambda) = \sup_{(\kappa,\lambda)\in\Lambda} \inf_{(U,D)\in\mathcal{A}} J_x(U,D,\kappa,\lambda) = J_x(U^*,D^*,\kappa^*,\lambda^*),$$

State of the art: ergodic singular control

- **Diffusions (no ambiguity)**
  Bast literature; early formulations for Brownian dynamics[1], broad settings[2,3,4], verification, and vanishing-discount approaches[5,6,7].
- **Diffusions with jumps (no ambiguity)**
  Scarcer literature and largely intractable; successful attempt[8] using a Dynkin-game representation and vanishing discount, other works[9,10,11,12] treating discounted case.
- **Model ambiguity (diffusions)**
  Very limited literature; zero-sum game and (max-min) HJB characterization[13]; mean-field (many controllers) extension[14].
- **Model ambiguity with jumps**
  We are not aware of results in this category.

[1] Karatzas (1983). A class of singular stochastic control problems. *Advances in Applied Probability*
[2] Menaldi et al. (1984). Some singular control problem with long term average criterion. In: Springer
[3] Weerasinghe (2002). Stationary stochastic control for Itô processes. *Advances in Applied Probability*
[4] Kunwai et al. (2022). On an Ergodic Two-Sided Singular Control Problem. *Applied Mathematics & Optimization*
[5] Weerasinghe (2007). An Abelian Limit Approach to a Singular Ergodic Control Problem. *SIAM Journal on Control and Optimization*
[6] Arapostathis et al. (2011). Ergodic Control of Diffusion Processes. Cambridge University Press
[7] Menaldi et al. (2013). Singular ergodic control for multidimensional Gaussian–Poisson processes. *Stochastics*
[8] Mordecki et al. (July 2025). Two sided long-time optimization singular control problems for Lévy processes and Dynkin's games.
[9] Yamazaki (2017). Inventory Control for Spectrally Positive Lévy Demand Processes. *Mathematics of Operations Research*
[10] Hernández-Hernández et al. (2016). Optimality of Refraction Strategies for Spectrally Negative Lévy Processes. *SIAM Journal on Control and Optimization*
[11] Pérez et al. (2020). Optimal Periodic Replenishment Policies for Spectrally Positive Lévy Demand Processes. *SIAM Journal on Control and Optimization*
[12] Noba et al. (2023). On Singular Control for Lévy Processes. *Mathematics of Operations Research*. Publisher: INFORMS
[13] Cohen et al. (2022). Optimal Ergodic Harvesting under Ambiguity. *SIAM Journal on Control and Optimization*
[14] Ferrari et al. (May 2025). Stationary Mean-Field Games of Singular Control under Knightian Uncertainty.

The HJB equation

The dynamic programming principle suggests the existence of a function $V:\mathbb{R}\to\mathbb{R}$ and a constant γ satisfying the HJB equation

$$\min\left\{c_D - V'(x), c_U + V'(x), c(x) + \sup_{(\kappa,\lambda)\in\Lambda}\left\{(\mathbb{L}^{\kappa,\lambda}V)(x)\right\} - \gamma\right\} = 0.$$

- **$\mathbb{P}^{\kappa,\lambda}$-generator of X_t:**

$$(\mathbb{L}^{\kappa,\lambda}f)(x) = (b(x) + \sigma\kappa(x) - r\,\mathbb{E}\,[Y])f'(x) + \frac{1}{2}\sigma^2 f''(x) + \lambda(x)(\Psi f)(x),$$

  for $(\Psi f)(x) = \int_{\mathbb{R}}(f(x+y) - f(x))\,\mathrm{d}F(y)$.

- **Bang-bang ambiguity regions:** from the HJB, the optimal ambiguity functions (those where $\sup_{(\kappa,\lambda)\in\Lambda}\left\{(\mathbb{L}^{\kappa,\lambda}V)(x)\right\}$ is attained) are given by

$$\kappa^*(x) := \begin{cases}\delta, & V'(x) \geqslant 0\\ -\delta, & V'(x) < 0\end{cases}, \quad \lambda^*(x) := \begin{cases} r(1+\varepsilon), & (\Psi V)(x) \geqslant 0\\ r(1-\varepsilon), & (\Psi V)(x) < 0\end{cases}$$

- **Constant-barrier Skorokhod reflections:**
  A pair $(U^{\underline{x}}, D^{\overline{x}})$ for constants $\underline{x} < \overline{x}$, such that:
  - $\overline{X} := X_t^{U^{\underline{x}}, D^{\overline{x}}}$ **stays inside the band** $[\underline{x}, \overline{x}]$ for all ambiguity functions:

$$\overline{X} \in [\underline{x}, \overline{x}], \quad \mathbb{P}^{\kappa,\lambda}\text{-a.s. for all } (\kappa, \lambda) \in \Lambda,$$

  - **The Skorokhod complementary condition holds**:
    (it's never optimal pushing simultaneously from both sides)

$$\int_0^\infty (\overline{X} - \underline{x}) \, \mathrm{d}U_t^{\underline{x}} = \int_0^\infty (\overline{x} - \overline{X}) \, \mathrm{d}D_t^{\overline{x}} = 0.$$

Guess (CB-SR)

The optimal controls are among constant-barriers Skorokhod reflections:

$$(U^*, D^*) = (U^{\underline{x}}, D^{\overline{x}})$$

We use the CB-SR guess to **build and verify a candidate solution**.

Verification theorem

Suppose that $V : \mathbb{R} \to \mathbb{R}$ and $\gamma \in \mathbb{R}$ solve the free-boundary problem

$$\begin{cases} \mathbb{L}^* V(x) + c(x) = \gamma & x \in [\underline{x}, \overline{x}], & \text{(1a)} \\ \mathbb{L}^* V(x) + c(x) \geqslant \gamma & x \notin [\underline{x}, \overline{x}], & \text{(1b)} \\ -c_U \leqslant V'(x) \leqslant c_D, & x \in (\underline{x}, \overline{x}), & \text{(1c)} \\ V'(x) = -c_U, & x \leqslant \underline{x}, & \text{(1d)} \\ V'(x) = c_D, & x \geqslant \overline{x}, & \text{(1e)} \\ V \in C^2(\mathbb{R}), & & \text{(1f)} \end{cases}$$

for $\mathbb{L}^* = \mathbb{L}^{\kappa^*, \lambda^*}$. Then,

$$\gamma = \inf_{(U,D) \in \mathcal{A}} \sup_{(\kappa,\lambda) \in \Lambda} J_x(U, D, \kappa, \lambda) = \sup_{(\kappa,\lambda) \in \Lambda} \inf_{(U,D) \in \mathcal{A}} J_x(U, D, \kappa, \lambda) = J_x(U^*, D^*, \kappa^*, \lambda^*),$$

Negative jumps

Bang-bang singletons

If

- **FBP:** V satisfies the FBP (1a)-(1e);
- **Convexity guess:** V' increases in the inaction region $[\underline{x}, \overline{x}]$; (later verified)
- **Negative jumps:** $\mathrm{supp}(F) = (-\infty, 0]$ and $\mathbb{P}(Y < 0) > 0$.

Then, κ^* and λ^* take the form

$$\kappa^*(x) = \begin{cases} \delta, & x \geqslant x^\kappa \\ -\delta, & x < x^\kappa \end{cases}, \qquad \lambda^*(x) = \begin{cases} r(1+\varepsilon), & x \leqslant x^\lambda \\ r(1-\varepsilon), & x > x^\lambda \end{cases}, \tag{2}$$

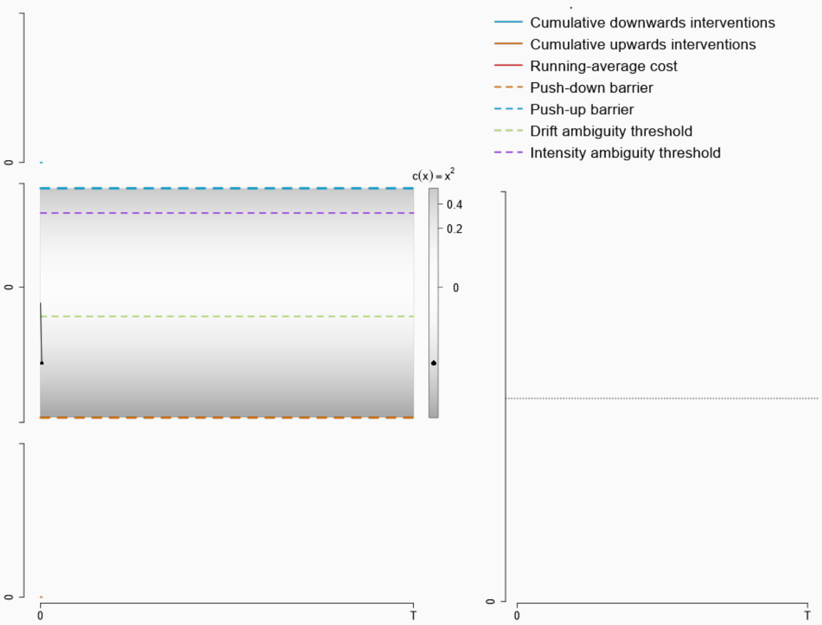
for unique values x^κ and x^λ such that one of these regimes hold

$$\text{Regime 1: } \underline{x} < x^\kappa < x^\lambda < \overline{x}, \qquad \text{Regime 2: } \underline{x} < x^\kappa < \overline{x} \leqslant x^\lambda.$$

Moreover,

$$\text{Regime 1} \iff \mathbb{E}\left[V(\overline{x} + Y) - V(\overline{x})\right] < 0.$$

Visualizing the Robust Ergodic Singular Problem (regime 1



Building a candidate solution

- **Fix boundaries:** Take $\underline{x}_*, \overline{x}_*, x_*^\kappa, x_*^\lambda \in \mathbb{R}$.
- **Define ambiguous functions:** κ^* and λ^* as in (2).
- **Free-boundary problem:** For $\mathbb{L}^* = \mathbb{L}^{\kappa^*,\lambda^*}$, assume that H_* solves the FBP

$$
\begin{cases}
(\mathbb{L}^* H_*)(x) + c'(x) = 0 & x \in (\underline{x}_*, \overline{x}_*) \setminus \{x_*^\kappa, x_*^\lambda\}, \quad (3a) \\
H_*(x) = -c_U, & x \leqslant \underline{x}_*, \quad (3b) \\
H_*(x) = c_D, & x \geqslant \overline{x}_*, \quad (3c) \\
H_*(x_*^\kappa) = 0, & (3d) \\
\int_{-\infty}^{0} H_*(x_*^\lambda + y) F(y) \, \mathrm{d}y = 0, & (3e) \\
H_* \in C^1(\mathbb{R}), & (3f) \\
H_* \in C^3\left((\underline{x}_*, \overline{x}_*) \setminus \{x_*^\kappa, x_*^\lambda\} \right), & (3g)
\end{cases}
$$

Candidate solution

We define our candidate solution V_* and γ_* such that

$$
V_*' = H_*, \qquad \gamma_* := -c_U(b - \delta\sigma - r\mathbb{E}[Y]) - c_U(1 + \varepsilon) r \mathbb{E}[Y] + c(\underline{x}_*).
$$

Assumptions

- **Negative jumps:** $\mathrm{supp}(F) = (-\infty, 0]$ and $\mathbb{P}(Y < 0) > 0$.
- **Constant drift:** $b(\cdot) \equiv b$.
- **Regular convex cost:** $c : \mathbb{R} \to \mathbb{R}$ is $C^2(\mathbb{R})$, convex everywhere and strictly convex in $[\underline{x}_*, \bar{x}_*]$, and it attains its unique minimum at $x = 0$.
- **Convexity of V_* at ambiguity thresholds:** $H_*'(x_*^\kappa) > 0, H_*'(x_*^\lambda) > 0$.

Verification theorem

Under the previous assumptions, the pair V_* and γ_* solve the FBP (1a)-(1f).

Example:
Negative-exponential jump-sizes and parabolic cost

Running assumptions

- **Negative-exponential jumps:** $F(y) = e^{\mu(y \wedge 0)}$, for all $y \in \mathbb{R}$, and for some $\mu > 0$.
- **Parabolic cost:** $c(x) = x^2$.
- **Constant drift:** $b(\cdot) \equiv b$.
- **Regime 1:** $\underline{x}_* < x_*^\kappa < x_*^\lambda < \overline{x}_*$. Regime 2 follows identical arguments.

Sufficient condition for Regime 1

The following condition suffice for Regime 1:

$$b \geqslant 0, \quad \varepsilon < c_D/(c_D + c_U), \quad (r/\mu)^{1/3} > \max\{2K_3/K_1, (2K_2/K_1)^{1/3}\},$$

for

$$K_1 := c_D - (c_D + c_U)\varepsilon, \quad K_2 := c_U(b + \delta\sigma) + 3((c_D + c_U)\sigma^2/4)^{2/3},$$
$$K_3 := 3((c_D + c_U)/4)^{2/3}(2(1+\varepsilon)/\mu)^{2/3}.$$

Candidate free-boundary problem

Candidate FBP

Under the running assumptions, there exists constants $\{a_{i,j}\}_{i,j=1}^{3}$ and $\{b_j\}_{j=0}^{2}$ such that, for

$$I_1^* = (\underline{x}_*, x_*^\kappa), \quad I_2^* := (x_*^\kappa, x_*^\lambda), \quad I_2^* := (x_*^\lambda, \overline{x}_*),$$

$H_* = V_*'$ and γ_* solves, in the inaction region, the piece-wise constant-coefficient ODE

$$0 = a_{i,3} H_*''(x) + a_{i,2} H_*'(x) + a_{i,1} H_*(x) + \mu x^2 + 2x - \mu \gamma_*, \quad x \in I_i^*,$$

while outside the inaction region,

$$H_* \equiv -c_U, \text{ on } (-\infty, \underline{x}_*], \qquad H_* \equiv c_D, \text{ on } [\overline{x}_*, \infty),$$

and the smooth-fit condition holds:

$$H_* \in C^1(\mathbb{R}).$$

$$0 = a_{i,3}H_*''(x) + a_{i,2}H_*'(x) + a_{i,1}H_*(x) + \mu x^2 + 2x - \mu\gamma_*$$

$$\Downarrow$$

ODE solution

$H \equiv H_i$ on each I_i^*, with

$$H_i(x) = \mathbf{c_i^-} e^{-\rho_i^- x} + \mathbf{c_i^+} e^{-\rho_i^+ x} + p_i(x, \gamma_*),$$

for the unknown coefficients $\mathbf{c_i^\pm}$ and the x-polynomial $p_i(x, \gamma_*)$ such that:

- $p_i(x)$ is of second degree iff $a_{i,1} \neq 0$;
- $p_i(x)$ is of third degree iff $a_{i,1} = 0$.

Exponents $\rho_i^\pm$ and polynomials' coefficients p_i are known and explicit from parameters. The polynomials p_i are linear in γ.

Unknowns and strategy

We need to find 11 unknowns

- **Ergodic value:** γ_*.
- **6 ODEs' constants:** $\{\mathbf{c}_\mathbf{i}^{\pm}\}_{i=1}^{3}$.
- **2 ambiguity thresholds:** x_*^{κ} and x_*^{λ}.
- **2 reflection barriers:** $\underline{x}_*$ and $\overline{x}_*$.

Strategy: two-stages solvers

Inner Solver

Compute γ and $\{\mathbf{c}_i^{\pm}\}_{i=1}^{3}$ provided $\mathbf{x} = (\underline{x}, x^{\kappa}, x^{\lambda}, \overline{x})$
Return $\{H_i\}_{i=1}^{3}$

Outer Solver

Compute C^1-residuals R and propose a new $\mathbf{x}$ to approach the root of R.

Inner Solver

- Obtain γ by solving $\mathbb{L}^{\kappa,\lambda}V(\underline{x}+) = 0$:

$$\gamma = \frac{1}{2}\sigma^2 H_1'(\underline{x},\gamma) + c_U(\delta\sigma - b - \varepsilon r\mathbb{E}[Y]) + c(\underline{x}).$$

- Obtain $\{\mathbf{c}_i^{\pm}\}_{i=1}^3$ by solving the $H \in C^0(\mathbb{R})$ linear system

$$\begin{cases} H_1(\underline{x}) = -c_U, \;\; H_3(\bar{x}) = c_D, \;\; H_1(x^\kappa) = H_2(x^\kappa), \;\; H_2(x^\lambda) = H_3(x^\lambda), \\ H_1(x^\kappa) = 0, \;\; \int_{-\infty}^{0} H(x^\lambda + y)\mu e^{\mu y}\,\mathrm{d}y = 0. \end{cases}$$

Outer Solver

Repeat 1-2 until $R \approx (0,0,0,0)$.

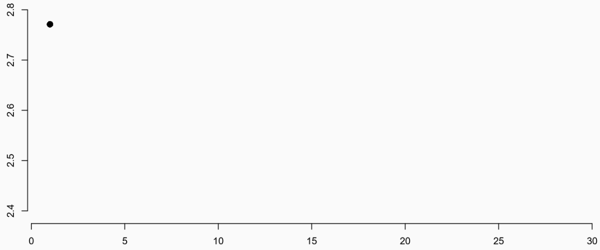
1. Given $\mathbf{x} = (\underline{x}, x^\kappa, x^\lambda, \bar{x})$, use Inner Solver to compute C^1 residuals:

$$R = \left(H_1'(\underline{x}),\; H_3'(\bar{x}),\; H_1'(x^\kappa) - H_2'(x^\kappa),\; H_2'(x^\lambda) - H_3'(x^\lambda)\right).$$

2. Propose a new $\mathbf{x}$ using a root-search updating algorithm (Broyden–Newton).

Numerics

Algorithm convergence



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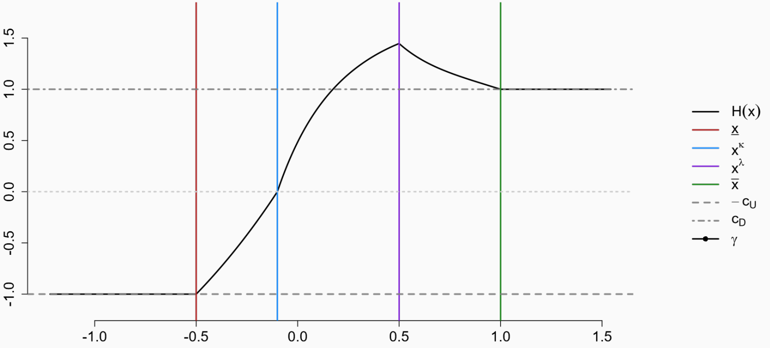
$H'(\underline{x}+) =$ 2.042e+00

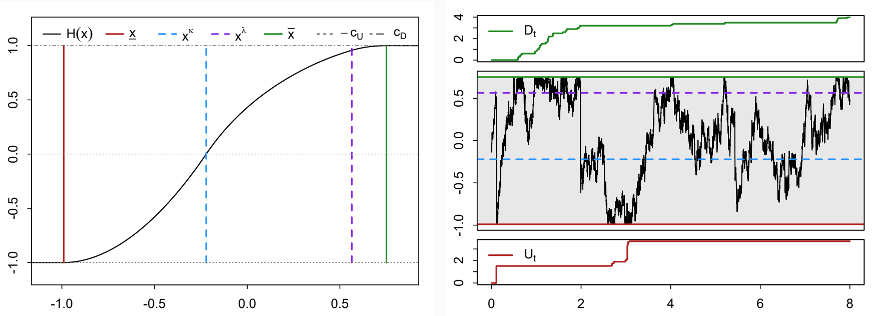
$\Delta H'(x^\kappa) =$ -2.838e+00

$\Delta H'(x^\lambda) =$ 2.588e+00

$H'(\overline{x}-) =$ -6.251e-01



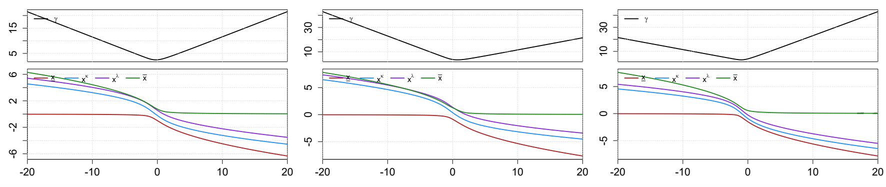
Baseline parameters: $b = 0$, $\delta = 1$, $r = 1$, $\varepsilon = 0.5$, $\sigma = 1$, and $\mu = 1$.



(a) H_* with reflecting barriers and ambiguity thresholds

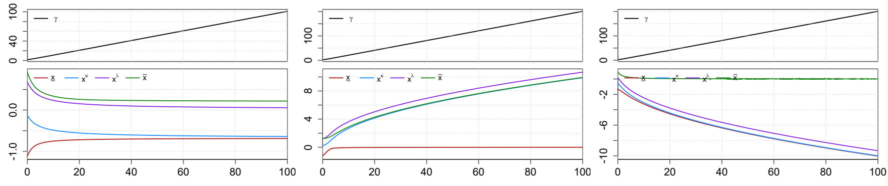
(b) Reflected path $\overline{X}$

Figure 1: Image (a): numerical computation of the function H_* with the corresponding ambiguity thresholds x^κ and x^λ, and the reflecting barriers $\underline{x}$ and $\overline{x}$. **Image (b)** controlled process $\overline{X}_t$. We used $c_U = 1$, $c_D = 1$.



(a) $c_U = 1, c_D = 1$ (b) $c_U = 2, c_D = 1$ (c) $c_U = 1, c_D = 2$

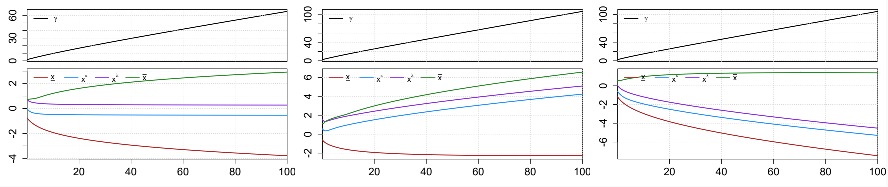
Figure 2: Reflecting barriers and ambiguity thresholds as ***b* varies**.



(a) $c_U = 1, c_D = 1$ (b) $c_U = 2, c_D = 1$ (c) $c_U = 1, c_D = 2$

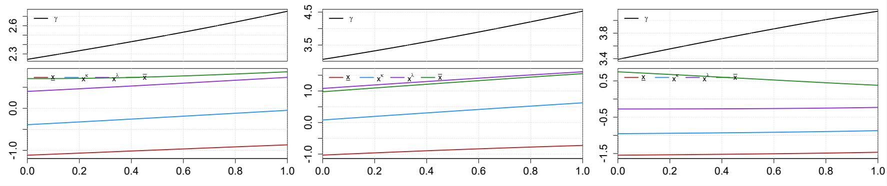
Figure 3: Reflecting barriers and ambiguity thresholds as **δ varies**.

COMPARATIVE STATISTICS. VARYING THE INTENSITY



(a) $c_U = 1, c_D = 1$ **(b)** $c_U = 2, c_D = 1$ **(c)** $c_U = 1, c_D = 2$

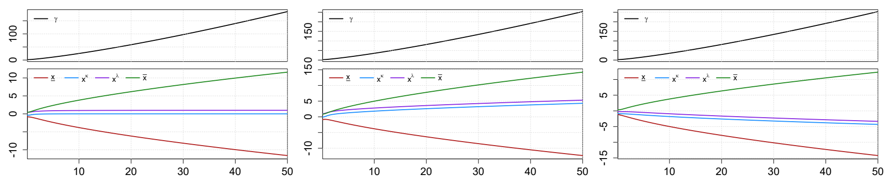
Figure 4: Reflecting barriers and ambiguity thresholds as ***r* varies**.



(a) $c_U = 1, c_D = 1$ **(b)** $c_U = 2, c_D = 1$ **(c)** $c_U = 1, c_D = 2$

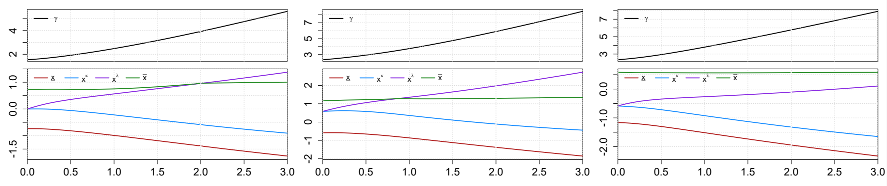
Figure 5: Reflecting barriers and ambiguity thresholds as ε **varies**.

Comparative statistics- Varying the diffusion and the jump-size mean



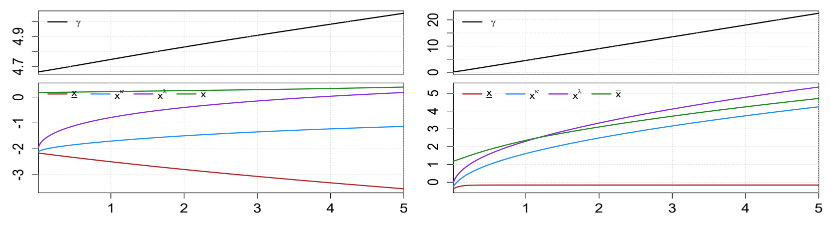
(a) $c_U = 1, c_D = 1$ **(b)** $c_U = 2, c_D = 1$ **(c)** $c_U = 1, c_D = 2$

Figure 6: Reflecting barriers and ambiguity thresholds as $\boldsymbol{\sigma}$ **varies**.



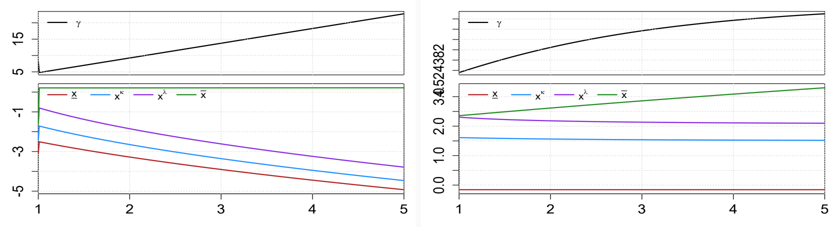
(a) $c_U = 1, c_D = 1$ **(b)** $c_U = 2, c_D = 1$ **(c)** $c_U = 1, c_D = 2$

Figure 7: Reflecting barriers and ambiguity thresholds as $-\mathbb{E}[Y] = 1/\mu$ **varies**.



(a) $b = 3$ **(b)** $b = -3$

Figure 8: Reflecting barriers and ambiguity thresholds as c_U varies.



(a) $b = 3$ **(b)** $b = -3$

Figure 9: Reflecting barriers and ambiguity thresholds as c_D varies.

Controller–adversary: the essential feedback loop

Controller (robust reflection)

- **Tolerate self-correcting moves**: relax where drift/jumps tend to pull back.
- **Block persistent risk**: tighten where worst-case dynamics can keep pushing into costly states.
- **Buffer the expensive action**: tilt the band toward the cheaper intervention.
- **Frequency vs exposure**: widen when volatility/shocks make reflection too frequent; shrink when adverse trends make exposure too costly.

Adversary (least-favorable switching)

- **Make damage stick**: select drift/intensity regimes that sustain deviations from low-cost regions.
- **Push toward costly reflection**: create one-sided pressure toward the boundary tied to the expensive control.
- **Remove helpful corrections**: downweight dynamics that would self-correct expensive excursions.

Gracias!